Optimization for Machine Learning

Lecture 3: Basic problems, Duality

6.881: MIT

Suvrit Sra
Massachusetts Institute of Technology

23 Feb, 2021

Basic convex problems

Linear Programming

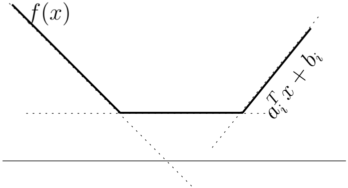
$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b, \quad Cx = d. \end{aligned}$$

Linear Programming

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b, \quad Cx = d. \end{aligned}$$

Piecewise linear minimization is an LP

$$\min \quad f(x) = \max_{1 \leq i \leq m} (a_i^T x + b_i)$$

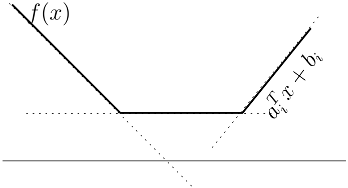


Linear Programming

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \le b, \quad Cx = d. \end{aligned}$$

Piecewise linear minimization is an LP

$$\min \quad f(x) = \max_{1 \le i \le m} (a_i^T x + b_i)$$



$$\min_{x,t} \quad t \qquad \text{s.t.} \quad a_i^T x + b_i \le t, \quad i = 1, \ldots, m.$$

Exercises

- Formulate $\min_x \|Ax - b\|_1$ as an LP ($\|x\|_1 = \sum_i |x_i|$)
- Formulate $\min_x \|Ax - b\|_\infty$ as an LP ($\|x\|_\infty = \max_{1 \le i \le n} |x_i|$)

Exercises

- Formulate $\min_x \|Ax - b\|_1$ as an LP ($\|x\|_1 = \sum_i |x_i|$)
- Formulate $\min_x \|Ax - b\|_\infty$ as an LP ($\|x\|_\infty = \max_{1 \le i \le n} |x_i|$)

Explore: LP formulations for Markov Decision Processes (MDPs). MDPs are frequently used models in Reinforcement Learning, and in some cases admit nice LP formulations.

Exercises

- Formulate $\min_x \|Ax - b\|_1$ as an LP ($\|x\|_1 = \sum_i |x_i|$)
- Formulate $\min_x \|Ax - b\|_\infty$ as an LP ($\|x\|_\infty = \max_{1\le i\le n} |x_i|$)

Explore: LP formulations for Markov Decision Processes (MDPs). MDPs are frequently used models in Reinforcement Learning, and in some cases admit nice LP formulations.

Explore: Integer LP: $\min_x c^T x, Ax \le b, x \in \mathbb{Z}^n$.

Exercises

- Formulate $\min_x \|Ax - b\|_1$ as an LP ($\|x\|_1 = \sum_i |x_i|$)
- Formulate $\min_x \|Ax - b\|_\infty$ as an LP ($\|x\|_\infty = \max_{1 \le i \le n} |x_i|$)

Explore: LP formulations for Markov Decision Processes (MDPs). MDPs are frequently used models in Reinforcement Learning, and in some cases admit nice LP formulations.

Explore: Integer LP: $\min_x c^T x, Ax \le b, x \in \mathbb{Z}^n$.

Open Problem. Can we solve the system of inequalities $Ax \le b$ in strongly polynomial time in the dimensions of the system, independent of the magnitudes of the coefficients? Best known result (Tardos, 1984) depends on coefficients of A but permits indpendence on magnitudes of b and the cost vector c.
N. Meggido, *On the complexity of linear programing*: Click here!

Quadratic Programming

$$\min \quad \frac{1}{2}x^T A x + b^T x + c \qquad \text{s.t. } Gx \leq h.$$

We assume $A \succeq 0$ (semidefinite).

Quadratic Programming

$$\min \quad \frac{1}{2}x^T A x + b^T x + c \qquad \text{s.t. } Gx \leq h.$$

We assume $A \succeq 0$ (semidefinite).

Exercise: Suppose no constraints; does QP always have solutions?

Quadratic Programming

$$\min \quad \tfrac{1}{2}x^T A x + b^T x + c \qquad \text{s.t. } Gx \leq h.$$

We assume $A \succeq 0$ (semidefinite).

Exercise: Suppose no constraints; does QP always have solutions?

Nonnegative least squares (NNLS)

$$\min \quad \tfrac{1}{2}\|Ax - b\|^2 \quad \text{s.t. } x \geq 0.$$

Exercise: Prove that NNLS always has a solution.

Regularized least-squares

Lasso

$$\min \quad \tfrac{1}{2}\|Ax - b\|_2^2 + \lambda\|x\|_1.$$

Exercise: How large must $\lambda > 0$ so that $x = 0$ is the optimum?

Regularized least-squares

Lasso

$$\min \quad \tfrac{1}{2}\|Ax - b\|_2^2 + \lambda\|x\|_1.$$

Exercise: How large must $\lambda > 0$ so that $x = 0$ is the optimum?

Total-variation denoising

$$\min \quad \tfrac{1}{2}\|Ax - b\|_2^2 + \lambda \sum_{i=1}^{n-1} |x_{i+1} - x_i|.$$

Exercise: Is the total-variation term a norm? Prove or disprove.

Regularized least-squares

Lasso

$$\min \quad \tfrac{1}{2}\|Ax - b\|_2^2 + \lambda\|x\|_1.$$

Exercise: How large must $\lambda > 0$ so that $x = 0$ is the optimum?

Total-variation denoising

$$\min \quad \tfrac{1}{2}\|Ax - b\|_2^2 + \lambda \sum\nolimits_{i=1}^{n-1} |x_{i+1} - x_i|.$$

Exercise: Is the total-variation term a norm? Prove or disprove.

Group Lasso

$$\min_{x_1,\ldots,x_T} \quad \frac{1}{2}\left\|b - \sum\nolimits_{j=1}^{T} A_j x_j\right\|_2^2 + \lambda \sum\nolimits_{j=1}^{T} \|x_j\|_2.$$

Exercise: What is the dual norm of the regularizer above?

Robust LP as an SOCP

$$\min \quad c^T x, \qquad \text{s.t.} \;\; a_i^T x \le b_i \;\; \forall a_i \in \mathcal{E}_i$$
$$\mathcal{E}_i := \{\bar{a}_i + P_i u \mid \|u\|_2 \le 1\}$$

Constraints are uncertain but with bounded uncertainty.

Robust LP as an SOCP

$$\min \quad c^T x, \qquad \text{s.t. } a_i^T x \leq b_i \;\; \forall a_i \in \mathcal{E}_i$$

$$\mathcal{E}_i := \{\bar{a}_i + P_i u \mid \|u\|_2 \leq 1\}$$

Constraints are uncertain but with bounded uncertainty.

(Adversarially) Robust LP formulation

$$\min_x \max_{\|u\|_2 \leq 1} \left\{ c^T x \mid a_i^T x \leq b_i, \quad a_i \in \mathcal{E}_i \right\}$$

Robust LP as an SOCP

$$\min \quad c^T x, \qquad \text{s.t. } a_i^T x \le b_i \ \ \forall a_i \in \mathcal{E}_i$$
$$\mathcal{E}_i := \{\bar{a}_i + P_i u \mid \|u\|_2 \le 1\}$$

Constraints are uncertain but with bounded uncertainty.

(Adversarially) Robust LP formulation

$$\min_x \max_{\|u\|_2 \le 1} \left\{ c^T x \mid a_i^T x \le b_i, \quad a_i \in \mathcal{E}_i \right\}$$

Second Order Cone Program

$$\min \quad c^T x, \qquad \text{s.t. } \|P_i^T x\|_2 \le -\bar{a}_i^T x + b_i, i = 1, \ldots, m.$$

SOCP constraint comes from:

$$\max_{\|u\|_2 \le 1} (\bar{a}_i + P_i u)^T x = \bar{a}_i^T x + \|P_i^T x\|_2$$

Exercise: Give a quick argument for above equality.

Semidefinite Program (SDP)

$$\min_{x \in \mathbb{R}^n} \quad c^T x$$
$$\text{s.t.} \quad A(x) := A_0 + x_1 A_1 + x_2 A_2 + \ldots + x_n A_n \succeq 0.$$

Semidefinite Program (SDP)

$$\min_{x \in \mathbb{R}^n} \quad c^T x$$
$$\text{s.t.} \quad A(x) := A_0 + x_1 A_1 + x_2 A_2 + \ldots + x_n A_n \succeq 0.$$

- $A_0, \ldots, A_n$ are real, symmetric matrices
- Inequality $A \preceq B$ means $B - A$ is *semidefinite*
- Also a cone program (conic optimization problem)

Semidefinite Program (SDP)

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^T x \\ \text{s.t.} \quad & A(x) := A_0 + x_1 A_1 + x_2 A_2 + \ldots + x_n A_n \succeq 0. \end{aligned}$$

- $A_0, \ldots, A_n$ are real, symmetric matrices
- Inequality $A \preceq B$ means $B - A$ is *semidefinite*
- Also a cone program (conic optimization problem)
- SDP $\supset$ SOCP $\supset$ QP $\supset$ LP
- **Exercise:** Write LPs, QPs, and SOCPs as SDPs

Semidefinite Program (SDP)

$$\min_{x \in \mathbb{R}^n} \quad c^T x$$
$$\text{s.t.} \quad A(x) := A_0 + x_1 A_1 + x_2 A_2 + \ldots + x_n A_n \succeq 0.$$

- $A_0, \ldots, A_n$ are real, symmetric matrices
- Inequality $A \preceq B$ means $B - A$ is *semidefinite*
- Also a cone program (conic optimization problem)
- SDP $\supset$ SOCP $\supset$ QP $\supset$ LP
- **Exercise:** Write LPs, QPs, and SOCPs as SDPs
- Feasible set of SDP is $\{$semidefinite cone $\bigcap$ hyperplanes$\}$

Explore: Which convex problems **representable** as SDPs? (This is an important topic in optimization theory).

Examples

♠ **Eigenvalue optimization:** $\min_x \ \lambda_{\max}(A(x))$

$$\min \quad t \quad \text{s.t.} \quad A(x) \preceq tI.$$

Examples

♠ **Eigenvalue optimization:** $\min_x \; \lambda_{\max}(A(x))$

$$\min \quad t \quad \text{s.t.} \quad A(x) \preceq tI.$$

♠ **Norm minimization:** $\min_x \|A(x)\|$

$$\min \quad t \quad \text{s.t.} \quad \begin{bmatrix} tI & A(x)^T \\ A(x) & tI \end{bmatrix} \succeq 0.$$

Examples

♠ **Eigenvalue optimization:** $\min_x \ \lambda_{\max}(A(x))$

$$\min \quad t \quad \text{s.t.} \quad A(x) \preceq tI.$$

♠ **Norm minimization:** $\min_x \|A(x)\|$

$$\min \quad t \quad \text{s.t.} \quad \begin{bmatrix} tI & A(x)^T \\ A(x) & tI \end{bmatrix} \succeq 0.$$

♠ More examples – see CVX documentation and BV book

Explore: SDP relaxations of nonconvex probs: important technique, starting with MaxCut SDP (Goemans-Williamson).

Explore: Sum-of-squares (SOS) optimization, Lasserre hierarchy of relaxations; see also: *https://www.sumofsquares.org*

Duality

(Weak duality, strong duality)

Primal problem

Let $f_i : \mathbb{R}^n \to \mathbb{R}$ $(1 \le i \le m)$. Generic ***nonlinear program***

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & f_i(x) \le 0, \quad 1 \le i \le m, \\ & x \in \{\operatorname{dom} f \cap \operatorname{dom} f_1 \cdots \cap \operatorname{dom} f_m\}. \end{aligned} \tag{P}$$

Primal problem

Let $f_i : \mathbb{R}^n \to \mathbb{R}$ $(1 \le i \le m)$. Generic ***nonlinear program***

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & f_i(x) \le 0, \quad 1 \le i \le m, \\ & x \in \{\operatorname{dom} f \cap \operatorname{dom} f_1 \cdots \cap \operatorname{dom} f_m\}. \end{aligned} \tag{P}$$

Domain: The set $\mathcal{X} := \{\operatorname{dom} f \cap \operatorname{dom} f_1 \cdots \cap \operatorname{dom} f_m\}$

- We call (P) the ***primal problem***
- The variable x is the ***primal variable***

Primal problem

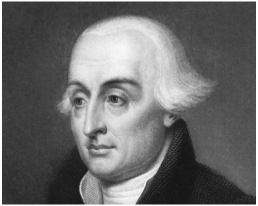
Let $f_i : \mathbb{R}^n \to \mathbb{R}$ $(1 \le i \le m)$. Generic ***nonlinear program***

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & f_i(x) \le 0, \quad 1 \le i \le m, \\ & x \in \{\operatorname{dom} f \cap \operatorname{dom} f_1 \cdots \cap \operatorname{dom} f_m\}. \end{aligned} \tag{P}$$

Domain: The set $\mathcal{X} := \{\operatorname{dom} f \cap \operatorname{dom} f_1 \cdots \cap \operatorname{dom} f_m\}$

- We call (P) the ***primal problem***
- The variable x is the ***primal variable***

Lagrangians and Duality



The reader will find no figures in this work. The methods which I set forth do not require either constructions or geometrical or mechanical reasonings: but only algebraic operations, subject to a regular and uniform rule of procedure.

—Joseph-Louis Lagrange
Preface to *Mécanique Analytique*

Lagrangian

To primal, associate ***Lagrangian*** $\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^m_+ \to (-\infty, \infty]$,

$$\mathcal{L}(x, \lambda) := f(x) + \sum_{i=1}^{m} \lambda_i f_i(x).$$

Lagrangian

To primal, associate ***Lagrangian*** $\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^m_+ \to (-\infty, \infty]$,

$$\mathcal{L}(x, \lambda) := f(x) + \sum\nolimits_{i=1}^m \lambda_i f_i(x).$$

♠ Variables $\lambda \in \mathbb{R}^m_+$ called ***Lagrange multipliers***

Lagrangian

To primal, associate *Lagrangian* $\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^m_+ \to (-\infty, \infty]$,

$$\mathcal{L}(x, \lambda) := f(x) + \sum\nolimits_{i=1}^{m} \lambda_i f_i(x).$$

- Variables $\lambda \in \mathbb{R}^m_+$ called *Lagrange multipliers*
- Suppose x feasible, and $\lambda \geq 0$. Lower-bound holds:

$$f(x) \geq \mathcal{L}(x, \lambda) \qquad \forall x \in \mathcal{X}, \ \ \lambda \in \mathbb{R}^m_+.$$

Lagrangian

To primal, associate ***Lagrangian*** $\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^m_+ \to (-\infty, \infty]$,

$$\mathcal{L}(x, \lambda) := f(x) + \sum_{i=1}^m \lambda_i f_i(x).$$

- Variables $\lambda \in \mathbb{R}^m_+$ called ***Lagrange multipliers***
- Suppose x feasible, and $\lambda \geq 0$. Lower-bound holds:

$$f(x) \geq \mathcal{L}(x, \lambda) \qquad \forall x \in \mathcal{X}, \;\; \lambda \in \mathbb{R}^m_+.$$

- In other words,

$$\sup_{\lambda \in \mathbb{R}^m_+} \mathcal{L}(x, \lambda) = \begin{cases} f(x), & \text{if } x \text{ feasible,} \\ +\infty & \text{otherwise.} \end{cases}$$

Proof on next slide

Lagrangian – proof

$$\mathcal{L}(x, \lambda) := f(x) + \sum_{i=1}^{m} \lambda_i f_i(x).$$

- $f(x) \geq \mathcal{L}(x, \lambda), \forall\, x \in \mathcal{X},\ \lambda \in \mathbb{R}^m_+$; so ***primal optimal*** (value)

$$p^* = \inf_{x \in \mathcal{X}} \sup_{\lambda \geq 0} \quad \mathcal{L}(x, \lambda).$$

Lagrangian – proof

$$\mathcal{L}(x, \lambda) := f(x) + \sum_{i=1}^{m} \lambda_i f_i(x).$$

- $f(x) \geq \mathcal{L}(x, \lambda), \forall\, x \in \mathcal{X},\ \lambda \in \mathbb{R}^m_+$; so ***primal optimal*** (value)

$$p^* = \inf_{x \in \mathcal{X}} \sup_{\lambda \geq 0} \quad \mathcal{L}(x, \lambda).$$

- If x is not feasible, then some $f_i(x) > 0$

Lagrangian – proof

$$\mathcal{L}(x,\lambda) := f(x) + \sum_{i=1}^m \lambda_i f_i(x).$$

- $f(x) \geq \mathcal{L}(x,\lambda), \forall\, x \in \mathcal{X},\ \lambda \in \mathbb{R}^m_+$; so ***primal optimal*** (value)

$$p^* = \inf_{x\in\mathcal{X}} \sup_{\lambda\geq 0} \quad \mathcal{L}(x,\lambda).$$

- If x is not feasible, then some $f_i(x) > 0$
- In this case, inner sup is $+\infty$, so claim true by definition

Lagrangian – proof

$$\mathcal{L}(x,\lambda) := f(x) + \textstyle\sum_{i=1}^m \lambda_i f_i(x).$$

- $f(x) \geq \mathcal{L}(x,\lambda), \forall\, x \in \mathcal{X},\ \lambda \in \mathbb{R}^m_+$; so *primal optimal* (value)

$$p^* = \inf_{x\in\mathcal{X}} \sup_{\lambda\geq 0} \quad \mathcal{L}(x,\lambda).$$

- If x is not feasible, then some $f_i(x) > 0$
- In this case, inner sup is $+\infty$, so claim true by definition
- If x is feasible, each $f_i(x) \leq 0$, so $\sup_\lambda \sum_i \lambda_i f_i(x) = 0$

Dual value

$$\mathcal{L}(x, \lambda) := f(x) + \sum_{i=1}^{m} \lambda_i f_i(x).$$

Primal value $\in [-\infty, +\infty]$

$$p^* = \inf_{x \in \mathcal{X}} \sup_{\lambda \geq 0} \quad \mathcal{L}(x, \lambda).$$

Dual value

$$\mathcal{L}(x,\lambda) := f(x) + \sum_{i=1}^{m} \lambda_i f_i(x).$$

Primal value $\in [-\infty, +\infty]$

$$p^* = \inf_{x \in \mathcal{X}} \sup_{\lambda \geq 0} \quad \mathcal{L}(x,\lambda).$$

Dual value $\in [-\infty, +\infty]$

$$d^* = \sup_{\lambda \geq 0} \inf_{x \in \mathcal{X}} \quad \mathcal{L}(x,\lambda).$$

Dual value

$$\mathcal{L}(x,\lambda) := f(x) + \sum_{i=1}^m \lambda_i f_i(x).$$

Primal value $\in [-\infty, +\infty]$

$$p^* = \inf_{x\in\mathcal{X}} \sup_{\lambda\geq 0} \quad \mathcal{L}(x,\lambda).$$

Dual value $\in [-\infty, +\infty]$

$$d^* = \sup_{\lambda\geq 0} \inf_{x\in\mathcal{X}} \quad \mathcal{L}(x,\lambda).$$

Dual function

$$g(\lambda) := \inf_{x\in\mathcal{X}} \mathcal{L}(x,\lambda).$$

Observe that $g(\lambda)$ is always concave!

Weak duality theorem

Theorem. (Weak duality). $p^* \geq d^*$. (i.e., WD always holds)

Proof:

1. $f(x') \geq \mathcal{L}(x', \lambda) \quad \forall x' \in \mathcal{X}$
2. Thus, for any $x \in \mathcal{X}$, we have $f(x) \geq \inf_{x'} \mathcal{L}(x', \lambda) = g(\lambda)$
3. Now minimize over x on lhs to obtain

$$\forall\, \lambda \in \mathbb{R}^m_+ \qquad p^* \geq g(\lambda).$$

4. Thus, taking sup over $\lambda \in \mathbb{R}^m_+$ we obtain $p^* \geq d^*$.

Lagrangians - Exercise

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.}\; & f_i(x) \leq 0, \quad i = 1, \ldots, m, \\ & h_i(x) = 0, \quad i = 1, \ldots, p. \end{aligned}$$

Exercise: Show that we get the Lagrangian dual

$$g : \mathbb{R}_+^m \times \mathbb{R}^p : (\lambda, \nu) \mapsto \inf_x \quad \mathcal{L}(x, \lambda, \nu),$$

Lagrange variable ν corresponds to the equality constraints.

Exercise: Prove that $p^* \geq \sup_{\lambda \geq 0, \nu \in \mathbb{R}^p} g(\lambda, \nu) = d^*$.

Exercises: Some duals

Derive Lagrangian duals for the following problems

- Least-norm solution of linear equations: $\min x^T x$ s.t. $Ax = b$
- Dual of an LP
- Dual of an SOCP
- Dual of an SDP
- Study example (5.7) in BV (binary QP)

Strong duality

Duality gap

$$p^* - d^*$$

Duality gap

$$p^* - d^*$$

Strong duality holds if duality gap is zero: $p^* = d^*$

Duality gap

$$p^* - d^*$$

Strong duality holds if duality gap is zero: $p^* = d^*$

Several **sufficient** conditions known!

Duality gap

$$p^* - d^*$$

Strong duality holds if duality gap is zero: $p^* = d^*$

Several **sufficient** conditions known!

"Easy" necessary and sufficient conditions: unknown

Abstract duality gap theorem*

Theorem. Let $v : \mathbb{R}^m \to \mathbb{R}$ be the *primal value function*

$$v(u) := \inf \{f(x) \mid f_i(x) \leq u_i,\ 1 \leq i \leq m\}.$$

The following relations hold:

1. $p^* = v(0)$
2. $v^*(-\lambda) = \begin{cases} -g(\lambda) & \lambda \geq 0 \\ +\infty & \text{otherwise.} \end{cases}$
3. $d^* = v^{**}(0)$

So if $v(0) = v^{**}(0)$ we have strong duality

Remark: Conditions such as Slater's ensure $\partial v(0) \neq \emptyset$, which ensures v is finite and lsc at 0, whereby $v(0) = v^{**}(0)$ holds.

Slater's sufficient conditions

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.}\; & f_i(x) \leq 0, \quad 1 \leq i \leq m, \\ & Ax = b. \end{aligned}$$

Slater's sufficient conditions

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t. } & f_i(x) \leq 0, \quad 1 \leq i \leq m, \\ & Ax = b. \end{aligned}$$

Constraint qualification: There exists $x \in \operatorname{ri} \mathcal{X}$ s.t.

$$f_i(x) < 0, \qquad Ax = b.$$

In words: there is a **strictly feasible** point.

Slater's sufficient conditions

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.}\ & f_i(x) \leq 0, \quad 1 \leq i \leq m, \\ & Ax = b. \end{aligned}$$

Constraint qualification: There exists $x \in \operatorname{ri} \mathcal{X}$ s.t.

$$f_i(x) < 0, \qquad Ax = b.$$

In words: there is a **strictly feasible** point.

> **Theorem.** Let the primal problem be convex. If there is a point that is *strictly feasible* for the non-affine constraints (merely feasible for affine), then strong duality holds. Moreover, in this case, the dual optimal is attained (i.e., $\partial v(0) \neq \emptyset$).

See BV §5.3.2 for a proof; (above, v is the primal value function)

Example with positive duality-gap

$$\min_{x,y} e^{-x} \quad x^2/y \leq 0,$$

over the domain $\mathcal{X} = \{(x, y) \mid y > 0\}$.

Example with positive duality-gap

$$\min_{x,y} e^{-x} \quad x^2/y \leq 0,$$

over the domain $\mathcal{X} = \{(x,y) \mid y > 0\}$.
Clearly, only feasible $x = 0$. So $p^* = 1$

Example with positive duality-gap

$$\min_{x,y} e^{-x} \quad x^2/y \leq 0,$$

over the domain $\mathcal{X} = \{(x, y) \mid y > 0\}$.
Clearly, only feasible $x = 0$. So $p^* = 1$

$$\mathcal{L}(x, y, \lambda) = e^{-x} + \lambda x^2/y,$$

so dual function is

$$g(\lambda) = \inf_{x,y>0} e^{-x} + \lambda x^2 y = \begin{cases} 0 & \lambda \geq 0 \\ -\infty & \lambda < 0. \end{cases}$$

Example with positive duality-gap

$$\min_{x,y} e^{-x} \quad x^2/y \leq 0,$$

over the domain $\mathcal{X} = \{(x,y) \mid y > 0\}$.
Clearly, only feasible $x = 0$. So $p^* = 1$

$$\mathcal{L}(x,y,\lambda) = e^{-x} + \lambda x^2/y,$$

so dual function is

$$g(\lambda) = \inf_{x,y>0} e^{-x} + \lambda x^2 y = \begin{cases} 0 & \lambda \geq 0 \\ -\infty & \lambda < 0. \end{cases}$$

Dual problem

$$d^* = \max_{\lambda} 0 \qquad \text{s.t. } \lambda \geq 0.$$

Example with positive duality-gap

$$\min_{x,y} e^{-x} \quad x^2/y \le 0,$$

over the domain $\mathcal{X} = \{(x,y) \mid y > 0\}$.
Clearly, only feasible $x = 0$. So $p^* = 1$

$$\mathcal{L}(x,y,\lambda) = e^{-x} + \lambda x^2/y,$$

so dual function is

$$g(\lambda) = \inf_{x,y>0} e^{-x} + \lambda x^2 y = \begin{cases} 0 & \lambda \ge 0 \\ -\infty & \lambda < 0. \end{cases}$$

Dual problem

$$d^* = \max_{\lambda} 0 \qquad \text{s.t. } \lambda \ge 0.$$

Thus, $d^* = 0$, and gap is $p^* - d^* = 1$.

Example with positive duality-gap

$$\min_{x,y} e^{-x} \quad x^2/y \le 0,$$

over the domain $\mathcal{X} = \{(x, y) \mid y > 0\}$.
Clearly, only feasible $x = 0$. So $p^* = 1$

$$\mathcal{L}(x, y, \lambda) = e^{-x} + \lambda x^2/y,$$

so dual function is

$$g(\lambda) = \inf_{x,y>0} e^{-x} + \lambda x^2 y = \begin{cases} 0 & \lambda \ge 0 \\ -\infty & \lambda < 0. \end{cases}$$

Dual problem

$$d^* = \max_{\lambda} 0 \qquad \text{s.t. } \lambda \ge 0.$$

Thus, $d^* = 0$, and gap is $p^* - d^* = 1$.
Here, we had no strictly feasible solution.

Example: Support Vector Machine (SVM)

$$\begin{aligned} \min_{x,\xi} \quad & \tfrac{1}{2}\|x\|_2^2 + C\sum\nolimits_i \xi_i \\ \text{s.t.} \quad & Ax \geq 1 - \xi, \quad \xi \geq 0. \end{aligned}$$

Example: Support Vector Machine (SVM)

$$\begin{aligned} \min_{x,\xi} \quad & \tfrac{1}{2}\|x\|_2^2 + C\sum\nolimits_i \xi_i \\ \text{s.t.} \quad & Ax \geq 1 - \xi, \quad \xi \geq 0. \end{aligned}$$

$$L(x, \xi, \lambda, \nu) = \tfrac{1}{2}\|x\|_2^2 + C1^T\xi - \lambda^T(Ax - 1 + \xi) - \nu^T\xi$$

Example: Support Vector Machine (SVM)

$$\min_{x,\xi} \quad \tfrac{1}{2}\|x\|_2^2 + C\sum\nolimits_i \xi_i$$
$$\text{s.t.} \quad Ax \geq 1 - \xi, \quad \xi \geq 0.$$

$$L(x,\xi,\lambda,\nu) = \tfrac{1}{2}\|x\|_2^2 + C1^T\xi - \lambda^T(Ax - 1 + \xi) - \nu^T\xi$$

$$\begin{aligned} g(\lambda,\nu) &:= \inf L(x,\xi,\lambda,\nu) \\ &= \begin{cases} \lambda^T 1 - \tfrac{1}{2}\|A^T\lambda\|_2^2 & \lambda + \nu = C\mathbf{1} \\ +\infty & \text{otherwise} \end{cases} \\ d^* &= \max_{\lambda \geq 0, \nu \geq 0} \quad g(\lambda,\nu) \end{aligned}$$

Exercise: Using $\nu \geq 0$, eliminate ν from above dual and obtain the canonical ***dual SVM*** formulation.

Example: norm regularized problems

$$\min \quad f(x) + \|Ax\|$$

Example: norm regularized problems

$$\min \quad f(x) + \|Ax\|$$

Dual problem

$$\min_y \quad f^*(-A^T y) \quad \text{s.t. } \|y\|_* \leq 1.$$

Example: norm regularized problems

$$\min \quad f(x) + \|Ax\|$$

Dual problem

$$\min_y \quad f^*(-A^T y) \quad \text{s.t. } \|y\|_* \le 1.$$

Say $\|\bar{y}\|_* < 1$, such that $A^T\bar{y} \in \mathrm{ri}(\mathrm{dom}\, f^*)$, then we have strong duality—for instance if $0 \in \mathrm{ri}(\mathrm{dom}\, f^*)$

Exercise. Write the constrained form of *group-lasso*:

$$\min_{x_1,\ldots,x_T} \quad \frac{1}{2}\left\|b - \sum\nolimits_{j=1}^T A_j x_j\right\|_2^2 + \lambda \sum\nolimits_{j=1}^T \|x_j\|_2.$$

Example: Dual via Fenchel conjugates

$$\min_x f_0(x) \quad \text{s.t. } f_i(x) \le 0 \ \ (1 \le i \le m), \ \ Ax = b.$$

Introduce ν and λ as dual variables; consider Lagrangian

$$\mathcal{L}(x, \lambda, \nu) \quad := \quad f_0(x) + \sum\nolimits_i \lambda_i f_i(x) + \nu^T(Ax - b)$$

Example: Dual via Fenchel conjugates

$$\min_x f_0(x) \quad \text{s.t. } f_i(x) \le 0 \ \ (1 \le i \le m), \ \ Ax = b.$$

Introduce ν and λ as dual variables; consider Lagrangian

$$\begin{aligned} \mathcal{L}(x,\lambda,\nu) &:= f_0(x) + \sum\nolimits_i \lambda_i f_i(x) + \nu^T(Ax - b) \\ g(\lambda,\nu) &= \inf_x \mathcal{L}(x,\lambda,\nu) \end{aligned}$$

Example: Dual via Fenchel conjugates

$$\min_x f_0(x) \quad \text{s.t. } f_i(x) \le 0 \;\; (1 \le i \le m), \;\; Ax = b.$$

Introduce ν and λ as dual variables; consider Lagrangian

$$\begin{aligned}
\mathcal{L}(x, \lambda, \nu) &:= f_0(x) + \sum\nolimits_i \lambda_i f_i(x) + \nu^T(Ax - b) \\
g(\lambda, \nu) &= \inf_x \mathcal{L}(x, \lambda, \nu) \\
g(\lambda, \nu) &= -\nu^T b + \inf_x x^T A^T \nu + F(x)
\end{aligned}$$

Example: Dual via Fenchel conjugates

$$\min_x f_0(x) \quad \text{s.t. } f_i(x) \le 0 \;\; (1 \le i \le m), \;\; Ax = b.$$

Introduce ν and λ as dual variables; consider Lagrangian

$$\begin{aligned}
\mathcal{L}(x, \lambda, \nu) &:= f_0(x) + \sum\nolimits_i \lambda_i f_i(x) + \nu^T(Ax - b) \\
g(\lambda, \nu) &= \inf_x \mathcal{L}(x, \lambda, \nu) \\
g(\lambda, \nu) &= -\nu^T b + \inf_x x^T A^T \nu + F(x) \\
F(x) &:= f_0(x) + \sum\nolimits_i \lambda_i f_i(x)
\end{aligned}$$

Example: Dual via Fenchel conjugates

$$\min_x f_0(x) \quad \text{s.t. } f_i(x) \le 0 \;\; (1 \le i \le m), \;\; Ax = b.$$

Introduce ν and λ as dual variables; consider Lagrangian

$$\begin{aligned}
\mathcal{L}(x, \lambda, \nu) &:= f_0(x) + \sum\nolimits_i \lambda_i f_i(x) + \nu^T(Ax - b) \\
g(\lambda, \nu) &= \inf_x \mathcal{L}(x, \lambda, \nu) \\
g(\lambda, \nu) &= -\nu^T b + \inf_x x^T A^T \nu + F(x) \\
F(x) &:= f_0(x) + \sum\nolimits_i \lambda_i f_i(x) \\
g(\lambda, \nu) &= -\nu^T b - \sup_x \langle x, -A^T \nu \rangle - F(x)
\end{aligned}$$

Example: Dual via Fenchel conjugates

$$\min_x \ f_0(x) \quad \text{s.t. } f_i(x) \leq 0 \ \ (1 \leq i \leq m), \ \ Ax = b.$$

Introduce ν and λ as dual variables; consider Lagrangian

$$\begin{aligned}
\mathcal{L}(x, \lambda, \nu) &:= f_0(x) + \sum\nolimits_i \lambda_i f_i(x) + \nu^T(Ax - b) \\
g(\lambda, \nu) &= \inf_x \mathcal{L}(x, \lambda, \nu) \\
g(\lambda, \nu) &= -\nu^T b + \inf_x x^T A^T \nu + F(x) \\
F(x) &:= f_0(x) + \sum\nolimits_i \lambda_i f_i(x) \\
g(\lambda, \nu) &= -\nu^T b - \sup_x \langle x, \ -A^T \nu \rangle - F(x) \\
g(\lambda, \nu) &= -\nu^T b - F^*(-A^T \nu).
\end{aligned}$$

F^* seems rather opaque...

Example: Dual via Fenchel conjugates

Important trick: "variable splitting"

$$\min_x f_0(x) \quad \text{s.t.} \quad f_i(x_i) \leq 0, Ax = b$$
$$x = x_i, i = 1, \ldots, m.$$

Example: Dual via Fenchel conjugates

Important trick: "variable splitting"

$$\min_x f_0(x) \quad \text{s.t.} \quad f_i(x_i) \leq 0, Ax = b$$

$$x = x_i, i = 1, \ldots, m.$$

$$\mathcal{L}(x, x_i, \lambda, \nu, \pi_i)$$

$$:= f_0(x) + \sum\nolimits_i \lambda_i f_i(x_i) + \nu^T(Ax - b) + \sum\nolimits_i \pi_i^T(x_i - x)$$

Example: Dual via Fenchel conjugates

Important trick: "variable splitting"

$$\min_x f_0(x) \quad \text{s.t.} \quad f_i(x_i) \leq 0, Ax = b$$
$$x = x_i, i = 1, \ldots, m.$$

$$\mathcal{L}(x, x_i, \lambda, \nu, \pi_i)$$
$$:= f_0(x) + \sum\nolimits_i \lambda_i f_i(x_i) + \nu^T(Ax - b) + \sum\nolimits_i \pi_i^T(x_i - x)$$
$$g(\lambda, \nu, \pi_i) = \inf_{x, x_i} \mathcal{L}(x, x_i, \lambda, \nu, \pi_i)$$

Example: Dual via Fenchel conjugates

Important trick: "variable splitting"

$$\min_x f_0(x) \quad \text{s.t.} \qquad f_i(x_i) \le 0, Ax = b$$
$$x = x_i, i = 1, \ldots, m.$$

$$\mathcal{L}(x, x_i, \lambda, \nu, \pi_i)$$
$$:= f_0(x) + \sum\nolimits_i \lambda_i f_i(x_i) + \nu^T(Ax - b) + \sum\nolimits_i \pi_i^T(x_i - x)$$
$$g(\lambda, \nu, \pi_i) = \inf_{x, x_i} \mathcal{L}(x, x_i, \lambda, \nu, \pi_i)$$
$$= -\nu^T b + \inf_x \left(f_0(x) + \nu^T Ax - \sum\nolimits_i \pi_i^T x\right)$$
$$+ \sum\nolimits_i \inf_{x_i} \left(\pi_i^T x_i + \lambda_i f_i(x_i)\right),$$

Example: Dual via Fenchel conjugates

Important trick: "variable splitting"

$$\min_x f_0(x) \quad \text{s.t.} \qquad f_i(x_i) \leq 0, Ax = b$$
$$x = x_i, i = 1, \ldots, m.$$

$$\mathcal{L}(x, x_i, \lambda, \nu, \pi_i)$$
$$:= f_0(x) + \sum\nolimits_i \lambda_i f_i(x_i) + \nu^T(Ax - b) + \sum\nolimits_i \pi_i^T(x_i - x)$$
$$g(\lambda, \nu, \pi_i) = \inf_{x, x_i} \mathcal{L}(x, x_i, \lambda, \nu, \pi_i)$$
$$= -\nu^T b + \inf_x \left(f_0(x) + \nu^T A x - \sum\nolimits_i \pi_i^T x\right)$$
$$+ \sum\nolimits_i \inf_{x_i} \left(\pi_i^T x_i + \lambda_i f_i(x_i)\right),$$
$$= -\nu^T b - f^*\left(-A^T\nu + \sum\nolimits_i \pi_i\right) - \sum\nolimits_i (\lambda_i f_i)^*(-\pi_i).$$

(you may want to write $\sum_i \pi_i = s$)

Exercise: the variable splitting trick

$$\min_x \quad f(x) + h(x).$$

Exercise: Fill in the details for the following steps

$$\min_{x,z} \quad f(x) + h(z) \quad \text{s.t.} \quad x = z$$

$$L(x,z,\nu) = f(x) + h(z) + \nu^T(x - z)$$

$$g(\nu) = \inf_{x,z} L(x,z,\nu)$$

Strong duality: nonconvex example

Trust region subproblem (TRS)

$$\min \quad x^T A x + 2b^T x \qquad x^T x \le 1.$$

A is symmetric but not necessarily semidefinite!

Strong duality: nonconvex example

Trust region subproblem (TRS)

$$\min \quad x^T A x + 2b^T x \qquad x^T x \leq 1.$$

A is symmetric but not necessarily semidefinite!

Theorem. TRS always has zero duality gap.

Proof: Read Section 5.2.4 of BV.

See the challenge problems on pg 18, Lect1

von Neumann minmax theorem*

(**Simplified.**) Let A be linear, C, D be compact convex sets.

$$\min_{x \in C} \max_{y \in D} \langle Ax, y \rangle = \max_{y \in D} \min_{x \in C} \langle Ax, y \rangle.$$

von Neumann minmax theorem*

(**Simplified.**) Let A be linear, C, D be compact convex sets.

$$\min_{x \in C} \max_{y \in D} \langle Ax, y \rangle = \max_{y \in D} \min_{x \in C} \langle Ax, y \rangle.$$

von Neumann proved this via fixed-point theory. By considering the Fenchel problem

$$\min_{x} \quad \mathbb{1}_C(x) + \mathbb{1}_D^*(Ax),$$

we can conclude the theorem (some work required).