Optimization for Machine Learning

Lecture 12: Coordinate Descent, BCD, Altmin

6.881: MIT

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Coordinate descent

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Explore: Going through both $[n]$ and $[d]$?

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- Decide when/how to stop; *return* x^k

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- ♣ Renewed interest in CD was driven by ML
- ♣ Notice: in general CD is “derivative free”

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- ♣ **Cylic with permutation**: random order each cycle
- ♣ **Random sampling**: pick random index at each iteration

Which ones would you prefer? Why?

Exercise: CD for least squares

$$\min_x \; \|Ax - b\|_2^2$$

Exercise: Obtain an update for j-th coordinate

Coordinate descent update

$$x_j \leftarrow \frac{\sum_{i=1}^m a_{ij}\left(b_i - \sum_{l \neq j} a_{il} x_l\right)}{\sum_{i=1}^m a_{ij}^2}$$

(dropped superscripts, since we overwrite)

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- ♠ **Explore:** not easy to use for deep learning...

BCD

(Basics, Convergence)

Block coordinate descent (BCD)

$$\begin{aligned} \min \quad & f(\boldsymbol{x}) := f(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m) \\ & \boldsymbol{x} \in \mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \times \mathcal{X}_m. \end{aligned}$$

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Gauss-Seidel update

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Jacobi update (easy to parallelize)

$$\boldsymbol{x}_i^{k+1} \leftarrow \operatorname*{argmin}_{\boldsymbol{\xi} \in \mathcal{X}_i} f(\underbrace{\boldsymbol{x}_1^{k}, \ldots, \boldsymbol{x}_{i-1}^{k}}_{\text{don't clobber}}, \underbrace{\boldsymbol{\xi}}_{\text{current}}, \underbrace{\boldsymbol{x}_{i+1}^{k}, \ldots, \boldsymbol{x}_m^{k}}_{\text{todo}})$$

BCD – convergence

Theorem. Let f be C^1 over $\mathcal{X} := \prod_{i=1}^m \mathcal{X}_i$. Assume for each block i and $\boldsymbol{x} \in \mathcal{X}$, the minimum

$$\min_{\boldsymbol{\xi} \in \mathcal{X}_i} f(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{i+1}, \boldsymbol{\xi}, \boldsymbol{x}_{i+1}, \ldots, \boldsymbol{x}_m)$$

is **uniquely attained**. Then, every limit point of the sequence $\{x^k\}$ generated by BCD, is a stationary point of f.

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- **Unique solutions** of subproblems not always possible
- Above result is only **asymptotic** (holds in the limit)
- **Warning!** BCD may cycle indefinitely without converging, if number blocks > 2 and objective nonconvex.

BCD – Two blocks

Two block BCD

$$\text{minimize} f(\boldsymbol{x}) = f(\boldsymbol{x}_1, \boldsymbol{x}_2) \quad \boldsymbol{x} \in \mathcal{X}_1 \times \mathcal{X}_2.$$

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- No need of **unique solutions** to subproblems
- BCD for 2 blocks is also called: **Alternating Minimization**

CD – projection onto convex sets

$$\begin{aligned} \min \quad & \tfrac{1}{2}\|x-y\|_2^2 \\ \text{s.t.} \quad & x \in C_1 \cap C_2 \cap \cdots \cap C_m. \end{aligned}$$

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$$\min \quad \tfrac{1}{2}\|x - y\|_2^2 + \sum\nolimits_{i=1}^{m} \delta_{C_i}(x).$$

- ▶ Now invoke Douglas-Rachford using the product-space trick

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Solution 2: Take dual of the above formulation

Solution 1: Product space technique

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- New constraint: $x_1 = x_2 = \ldots = x_n$

$$\begin{aligned} \min_{(x_1,\ldots,x_n)} \quad & \sum\nolimits_i f_i(x_i) \\ \text{s.t.} \quad & x_1 = x_2 = \cdots = x_n. \end{aligned}$$

Technique due to: G. Pierra (1976)

Solution 1: Product space technique

$$\min_{\boldsymbol{x}} f(\boldsymbol{x}) + \mathbb{1}_{\mathcal{B}}(\boldsymbol{x})$$

where $\boldsymbol{x} \in \mathcal{H}^n$ and $\mathcal{B} = \{\boldsymbol{z} \in \mathcal{H}^n \mid \boldsymbol{z} = (x, x, \ldots, x)\}$

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$$\begin{aligned} \min\nolimits_{z \in \mathcal{B}} &\quad \tfrac{1}{2}\|\boldsymbol{z} - \boldsymbol{y}\|_2^2 \\ \min\nolimits_{x \in \mathcal{H}} &\quad \textstyle\sum_i \tfrac{1}{2}\|x - y_i\|_2^2 \\ \implies &\quad x = \tfrac{1}{n}\textstyle\sum_i y_i \end{aligned}$$

Exercise: Work out the details of the Douglas-Rachford algorithm using the above product space trick.

Remark: This technique commonly exploited in ADMM too

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$$L(x,z,w,\nu,\mu) := \tfrac{1}{2}\|x-y\|_2^2 + f(z) + h(w) + \nu^T(x-z) + \mu^T(x-w)$$

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Dual as minimization problem

$$\boxed{\min\ k(\nu,\mu) := \tfrac{1}{2}\|\nu+\mu-y\|_2^2 + f^*(\nu) + h^*(\mu)}$$

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Apply CD to $k(\nu, \mu) = \frac{1}{2}\|\nu + \mu - y\|_2^2 + f^*(\nu) + h^*(\mu)$

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  $\implies \nu = \operatorname{prox}_{f^*}(y - \mu_k) \implies \nu = y - \mu_k - \operatorname{prox}_f(y - \mu_k)$

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Apply CD to $k(\nu, \mu) = \frac{1}{2}\|\nu + \mu - y\|_2^2 + f^*(\nu) + h^*(\mu)$

$$\begin{aligned} \nu_{k+1} &= \operatorname{argmin}_\nu\ k(\nu, \mu_k) \\ \mu_{k+1} &= \operatorname{argmin}_\mu\ k(\nu_{k+1}, \mu) \end{aligned}$$

- $0 \in \nu + \mu_k - y + \partial f^*(\nu)$
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Proximal-Dykstra as CD

- Simplify, and use Lagrangian stationarity to obtain primal

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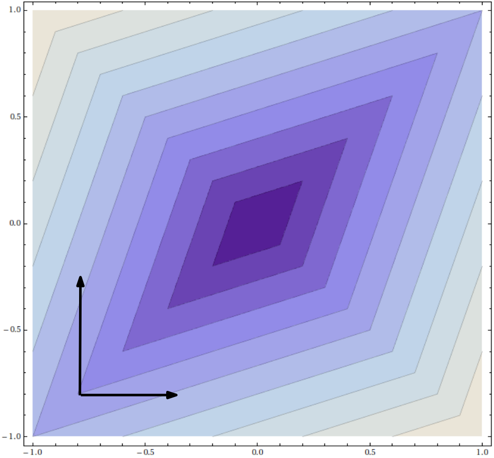
- We used: $\mathrm{prox}_h(y - \nu_{k+1}) = \mu_{k+1} - y - \nu_{k+1} = x_{k+1}$
- This is the proximal-Dykstra method!

Explore: Pros-cons of Prox-Dykstra versus product space+DR

CD – nonsmooth case

CD for nonsmooth convex problems

$$\min |x_1 - x_2| + \frac{1}{2}|x_1 + x_2|$$



CD for separable nonsmoothness

- Nonsmooth part is **separable**

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Theorem. If f is convex, continuously differentiable, each $r_i(x)$ is closed, convex, and each coordinate admits a **unique** solution. Further, assume we go through all coordinates in an essentially cyclic way. Then, the sequence $\{x^k\}$ generated by CD is bounded, and every limit point of it is optimal.

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Remark: A related result for **nonconvex** problems with separable non-smoothness (under more assumptions), can be found in: "*Convergence of Block Coordinate Descent Method for Nondifferentiable Minimization*" by P. Tseng (2001).

CD – iteration complexity

CD non-asymptotic rate

- So far, we saw CD based on essentially cyclic rules

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Here e_i denotes the ith canonical basis vector

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Choose $x_0 \in \mathbb{R}^d$. Let $M = \max_i L_i$; For $k \geq 0$

$$i_k = \underset{1\leq i\leq d}{\operatorname{argmax}} |\nabla_i f(x_k)|$$

$$x_{k+1} = x_k - \tfrac{1}{M}\nabla_{i_k} f(x_k) e_{i_k}.$$

CD – non-asymptotic convergence

Theorem. Let $\{x^k\}$ be iterate sequence generated by above greedy CD method. Then,

$$f(x_k) - f^* \leq \frac{2dM\|x_0 - x^*\|_2^2}{k+4}, \quad k \geq 0.$$

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- At each step, it requires access to **full gradient**
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- Also, if $f \in C_L^1$, it can easily happen that $M \geq L$
- So above rate is in general, worse than gradient methods

BCD – Notation

- **Decomposition:** $E = [E_1, \ldots, E_n]$ into n **blocks**
- Corresponding decomposition of x is

$$(\underbrace{E_1^T \boldsymbol{x}}_{N_1+}, \underbrace{E_2^T \boldsymbol{x}}_{N_2+}, \ldots, \underbrace{E_n^T \boldsymbol{x}}_{+N_n=N}) = (x^{(1)}, x^{(2)}, \ldots, x^{(n)})$$

- **Observation:**

$$E_i^T E_j = \begin{cases} I_{N_i} & i = j \\ 0_{N_i, N_j} & i \neq j. \end{cases}$$

- So the E_is define our partitioning of the coordinates
- Just fancier notation for a random partition of coordinates
- Now with this notation ...

BCD – formal setup

$$\boxed{\min\ f(\boldsymbol{x}) \text{ where } \boldsymbol{x} \in \mathbb{R}^d}$$

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- Using the descent lemma, we have blockwise upper bounds

$$f(\boldsymbol{x} + E_i h) \le f(\boldsymbol{x}) + \langle \nabla_i f(\boldsymbol{x}), h\rangle + \tfrac{L_i}{2}\|h\|^2, \quad \text{for } i = 1, \ldots, d.$$

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- At each step, minimize these upper bounds!

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We'll call this BCM (**Block Coordinate Minimization**)

Exercise: proximal extension

$$\min f(\boldsymbol{x}) + r(\boldsymbol{x})$$

- If **block separable** $r(\boldsymbol{x}) := \sum_{i=1}^{n} r_i(x^{(i)})$

$$x_k^{(i)} = \underset{h}{\operatorname{argmin}} f(\boldsymbol{x}_k) + \langle \nabla_i f(\boldsymbol{x}_k), h \rangle + \tfrac{L_i}{2}\|h\|^2 + r_i(E_i^T \boldsymbol{x}_k + h)$$

$$x_k^{(i)} = \operatorname{prox}_{r_i}(\cdots)$$

Exercise: Fill in the dots

$$h = \operatorname{prox}_{(1/L)r_i}\left(E_i^T \boldsymbol{x}_k - \tfrac{1}{L_i}\nabla_i f(\boldsymbol{x}_k)\right)$$

Randomized BCD – analysis

$$\boxed{h \leftarrow \operatorname{argmin}_h f(\boldsymbol{x}_k) + \langle \nabla_i f(\boldsymbol{x}_k), h \rangle + \tfrac{L_i}{2}\|h\|^2}$$

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f(\boldsymbol{x}_{k+1}) &\leq f(\boldsymbol{x}_k) - \tfrac{1}{2L_i}\|\nabla_i f(\boldsymbol{x}_k)\|^2.
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$$\boxed{f(\boldsymbol{x}_k) - f(\boldsymbol{x}_{k+1}) \geq \tfrac{1}{2L_i}\|\nabla_i f(\boldsymbol{x}_k)\|^2}$$

Randomized BCD – analysis

Expected descent:

$$f(\boldsymbol{x}_k) - \mathbb{E}[f(\boldsymbol{x}_{k+1}|\boldsymbol{x}_k)] \quad = \quad \sum_{i=1}^{d} p_i \big(f(\boldsymbol{x}_k) - f(\boldsymbol{x}_k - \tfrac{1}{L_i} E_i \nabla_i f(\boldsymbol{x}_k))\big)$$

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Descent plus some more (hard) work yields

$$
O\Big(\frac{d}{\epsilon} \sum\nolimits_i L_i \|x_0^{(i)} - x_*^{(i)}\|^2\Big)
$$

as the iteration complexity of obtaining $\mathbb{E}[f(\boldsymbol{x}_k)] - f^* \leq \epsilon$

BCD – Exercise

- Recall Lasso problem: $\min \frac{1}{2}\|Ax - b\|^2 + \lambda\|x\|_1$
- Here $x \in \mathbb{R}^N$
- Make $n = N$ blocks
- Show what the Randomized BCD iterations look like
- Notice, 1D prox operations for $\lambda|\cdot|$ arise
- Try to implement it as efficiently as you can (i.e., do not copy or update vectors / coordinates than necessary)

Connections

CD – exercise

$$\min \quad \frac{1}{n}\sum_{i=1}^{n} f_i(x^T a_i) + \frac{\lambda}{2}\|x\|^2.$$

Dual problem

$$\max_{\alpha} \quad \frac{1}{n}\sum_{i=1}^{n} -f_i^*(-\alpha_i) - \frac{\lambda}{2}\left\|\frac{1}{\lambda n}\sum_{i=1}^{n}\alpha_i a_i\right\|^2$$

Exercise: Study the SDCA algorithm and derive a connection between it and SAG/SAGA family of algorithms.

S. Shalev-Shwartz, T. Zhang. *Stochastic Dual Coordinate Ascent Methods for Regularized Loss Minimization*. JMLR (2013).

Other connections

Explore: Block-Coordinate Frank-Wolfe algorithm.

$$\min_x \quad f(x), \quad \text{s.t. } x \in \prod_i \mathcal{X}_i$$

Explore: Doubly stochastic methods

$$\min \quad f(x) = \sum\nolimits_i f_i(x_1, \ldots, x_d)$$

Being jointly stochastic over f_i as well as coordinates.

Explore: CD with constraints (linear and nonlinear constraints)