Locality-Sensitive Hashing and Beyond

Ilya Razenshteyn (MIT)

based on papers joint with

Alexandr Andoni (Columbia), Piotr Indyk (MIT),

Thijs Laarhoven (TU Eindhoven) and Ludwig Schmidt (MIT)

Outline

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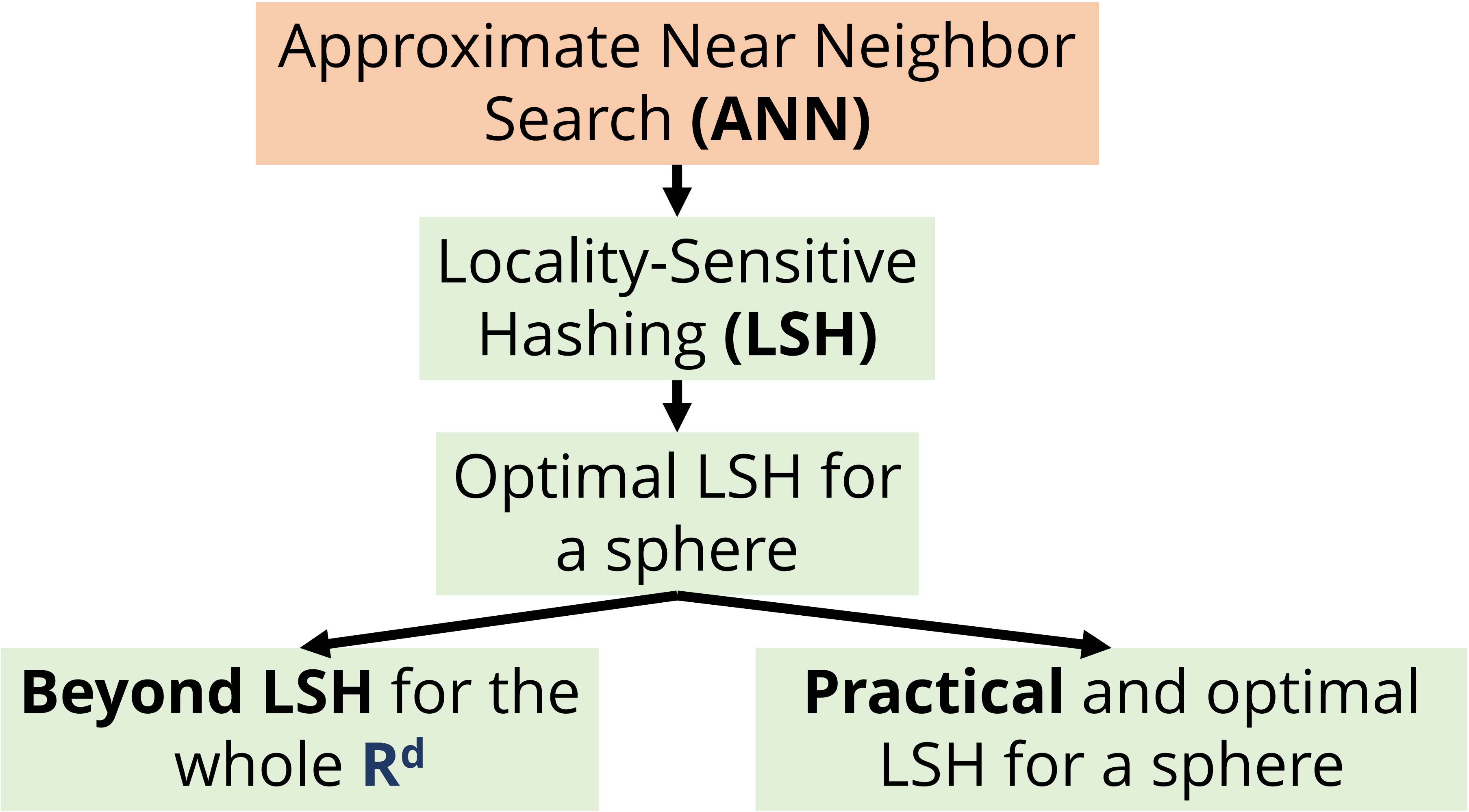
Approximate Near Neighbor Search **(ANN)**
Locality-Sensitive Hashing **(LSH)**
Optimal LSH for a sphere
Beyond LSH for the whole $\mathbf{R}^d$
Practical and optimal LSH for a sphere

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Approximate Near Neighbor Search **(ANN)**

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[Andoni, R 2015] STOC 2015

[Andoni, Indyk, Laarhoven, R, Schmidt 2015] NIPS 2015

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Near Neighbor Search

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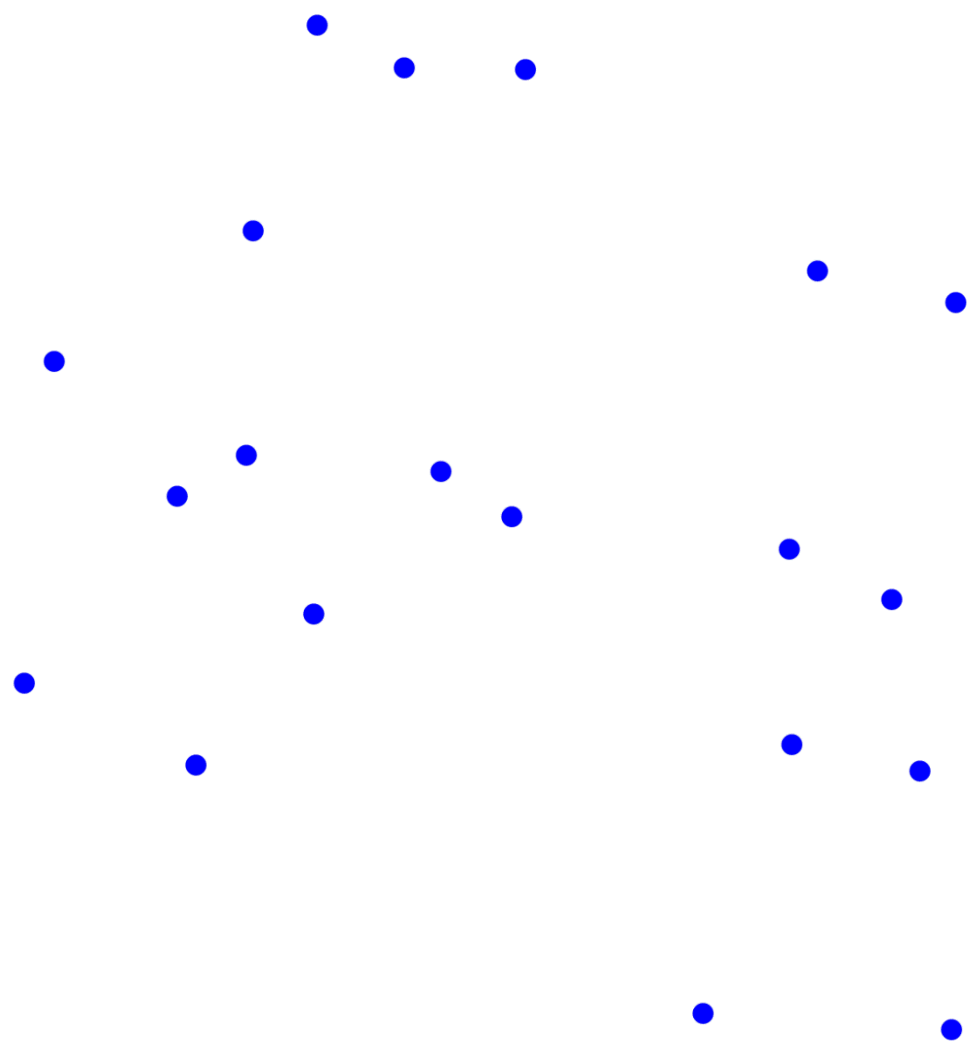

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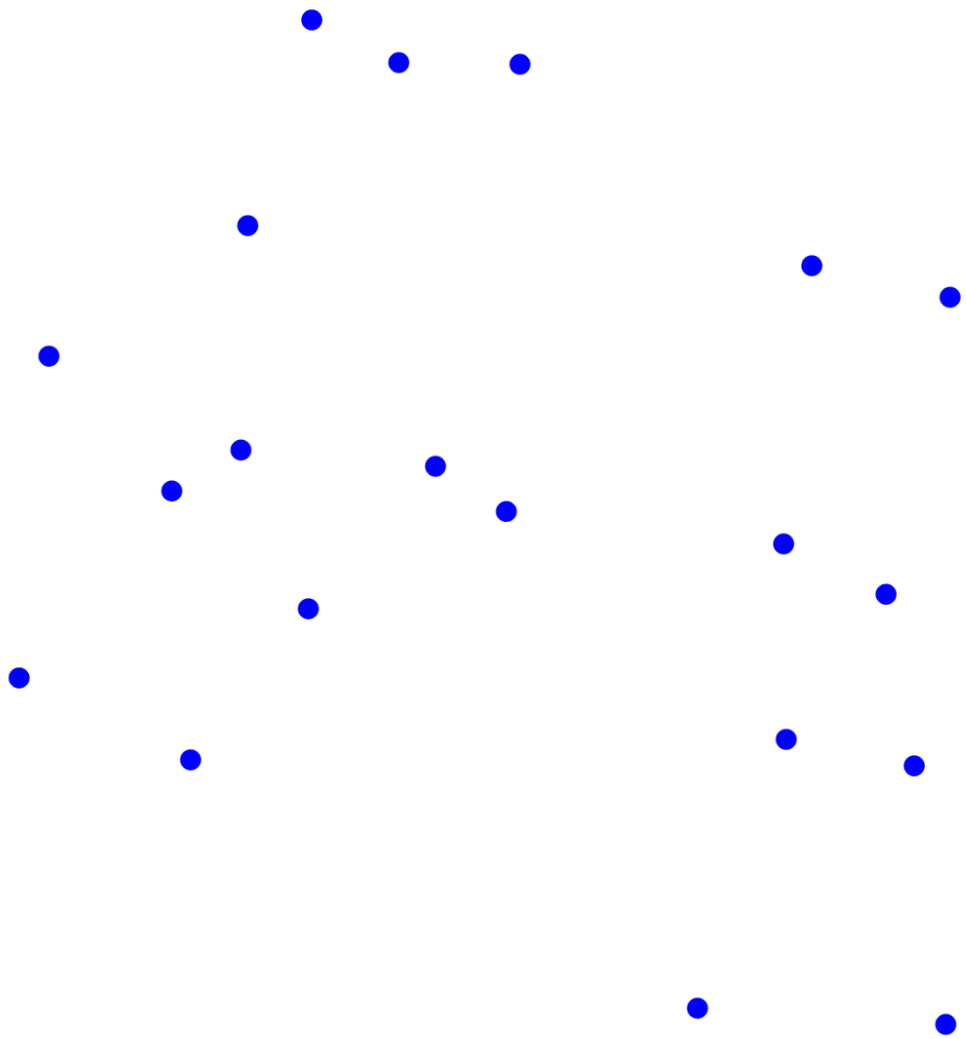

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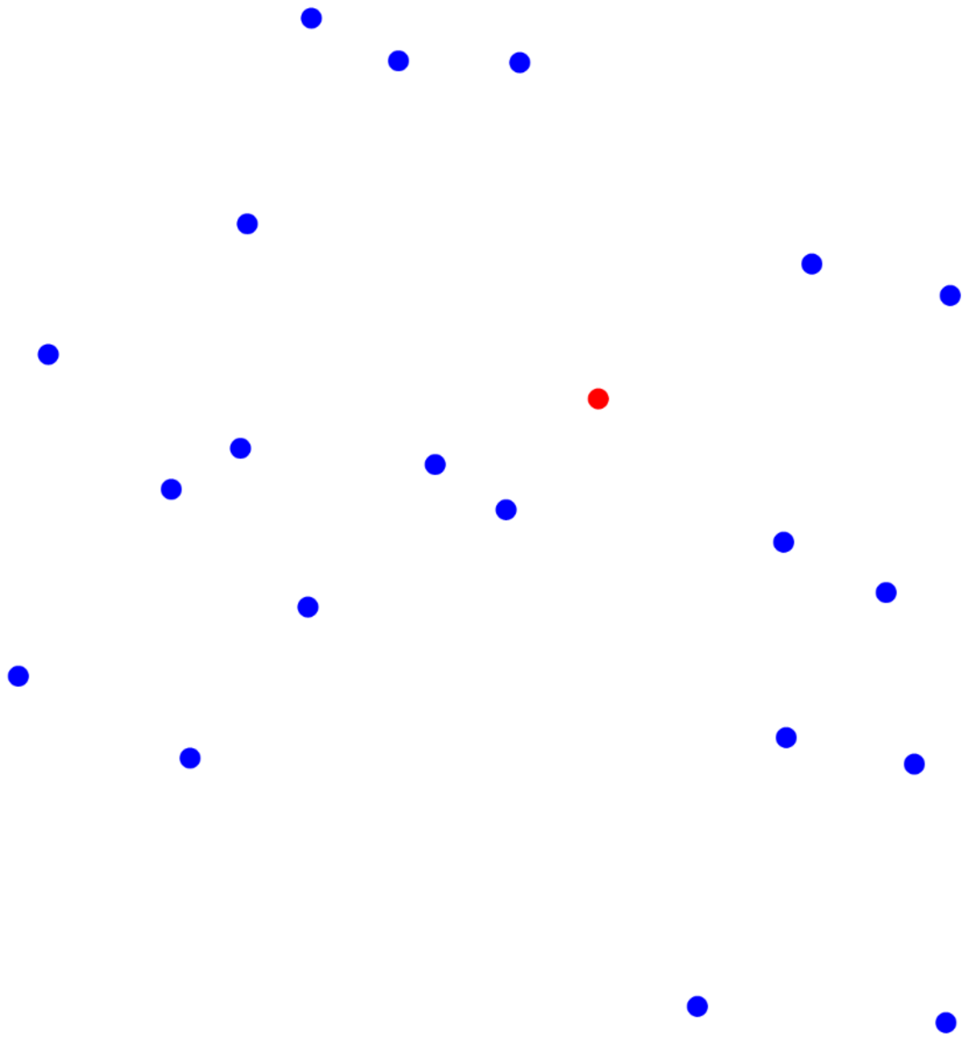

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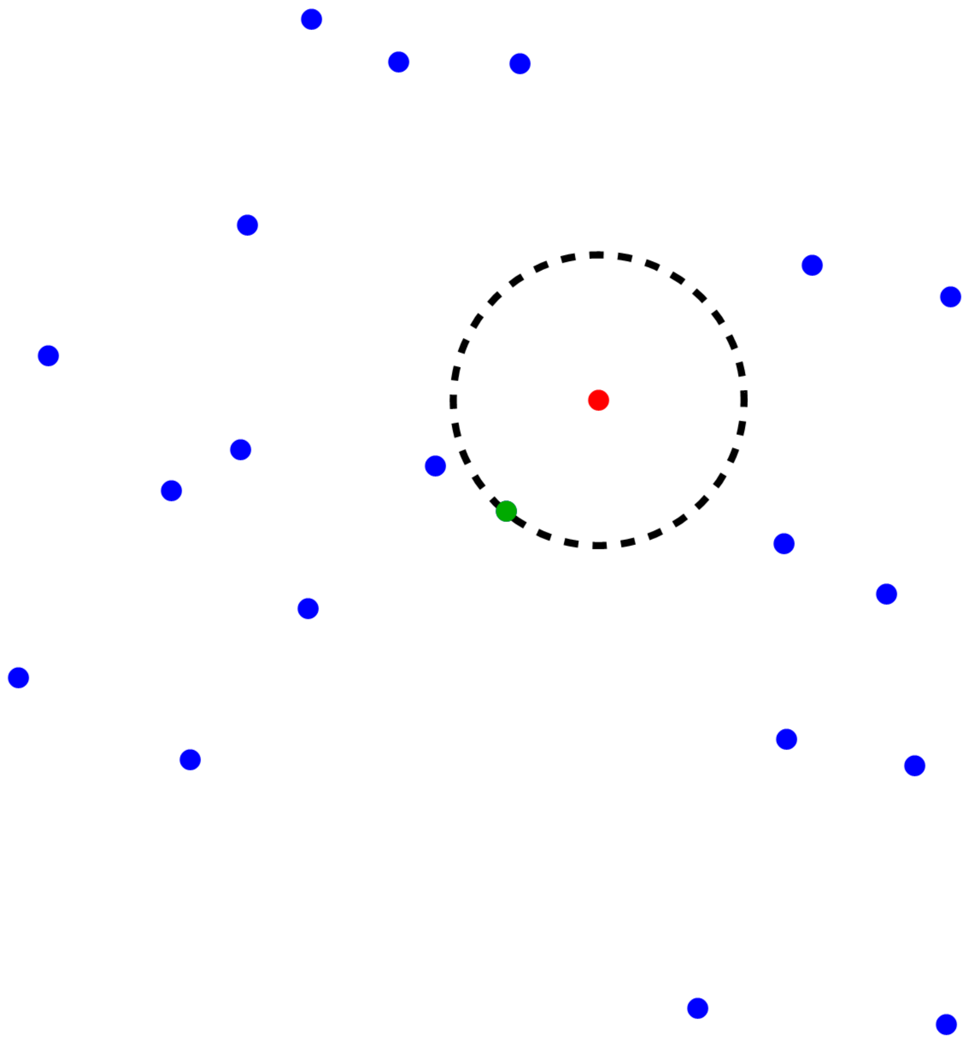

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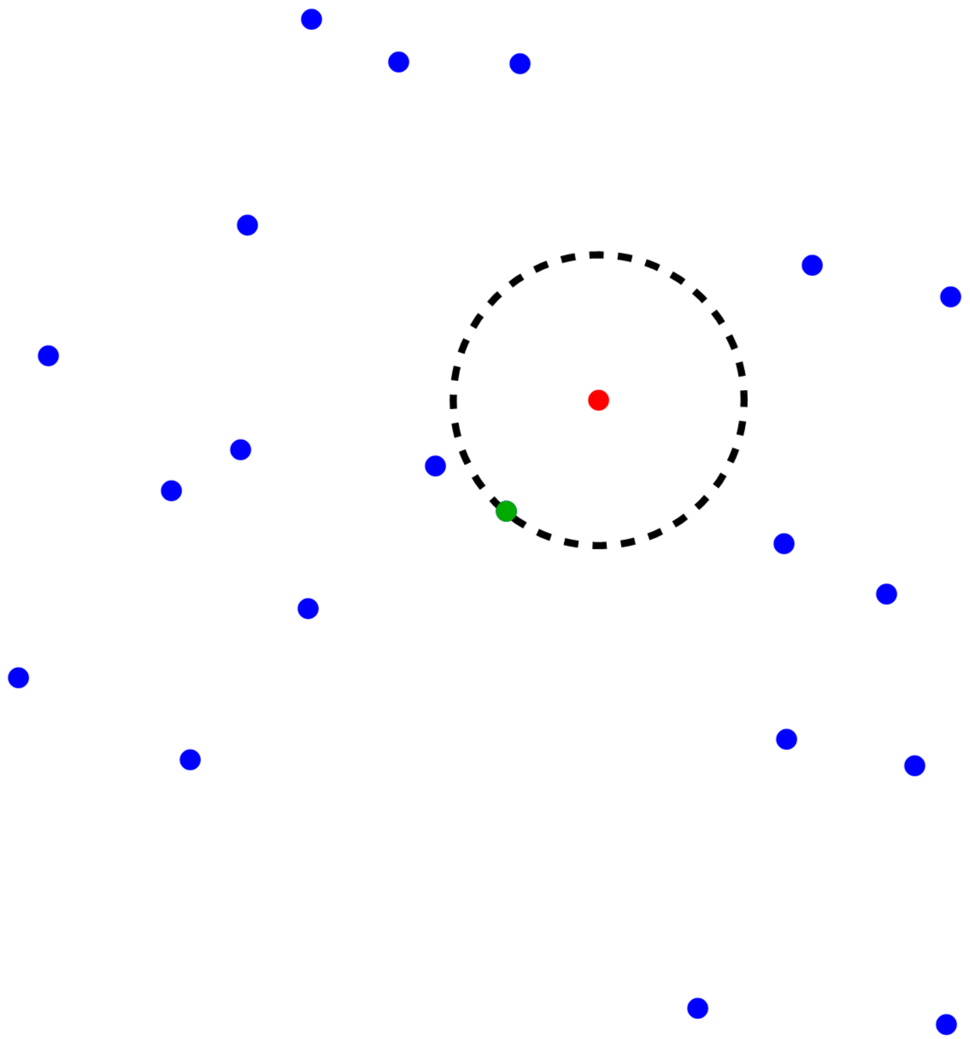

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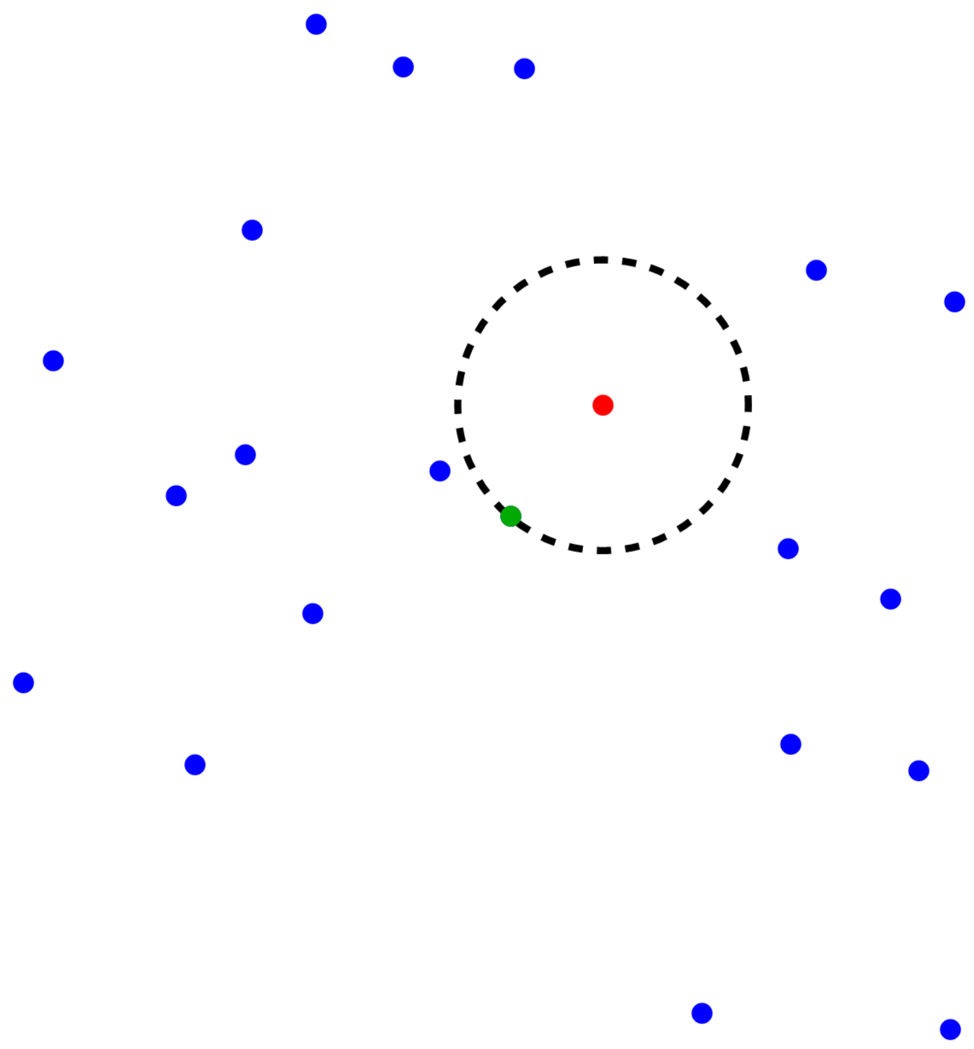

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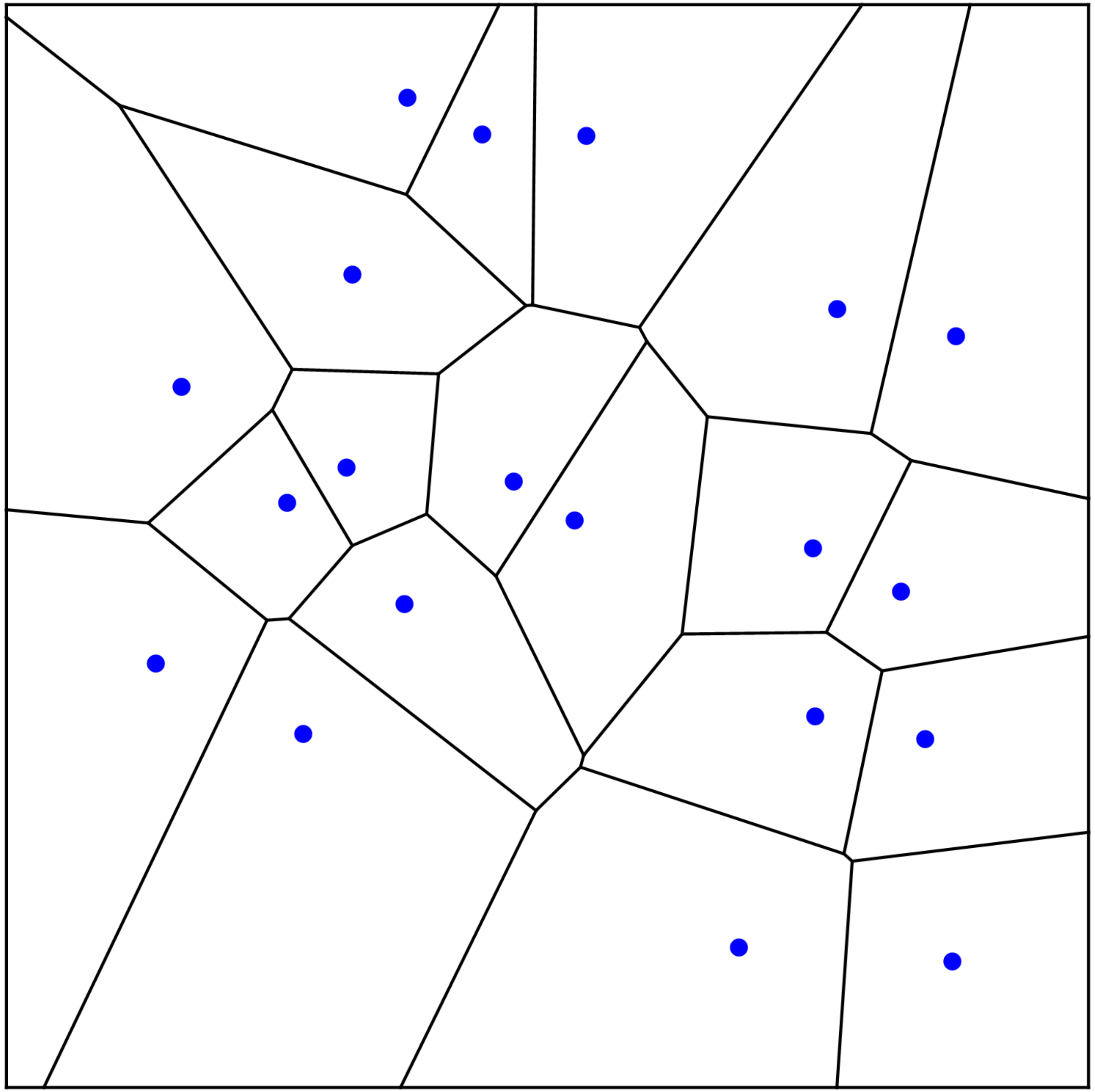

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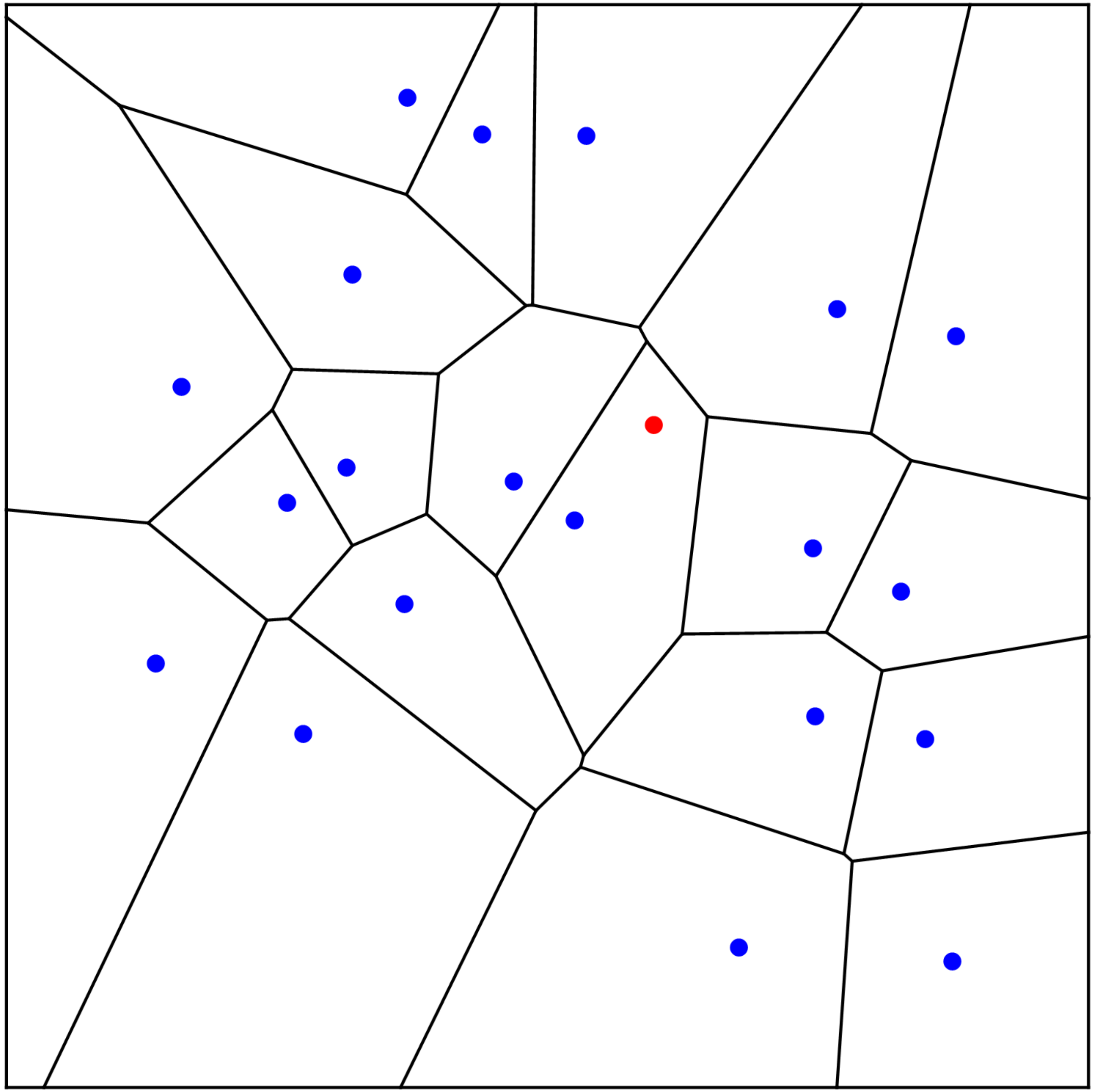

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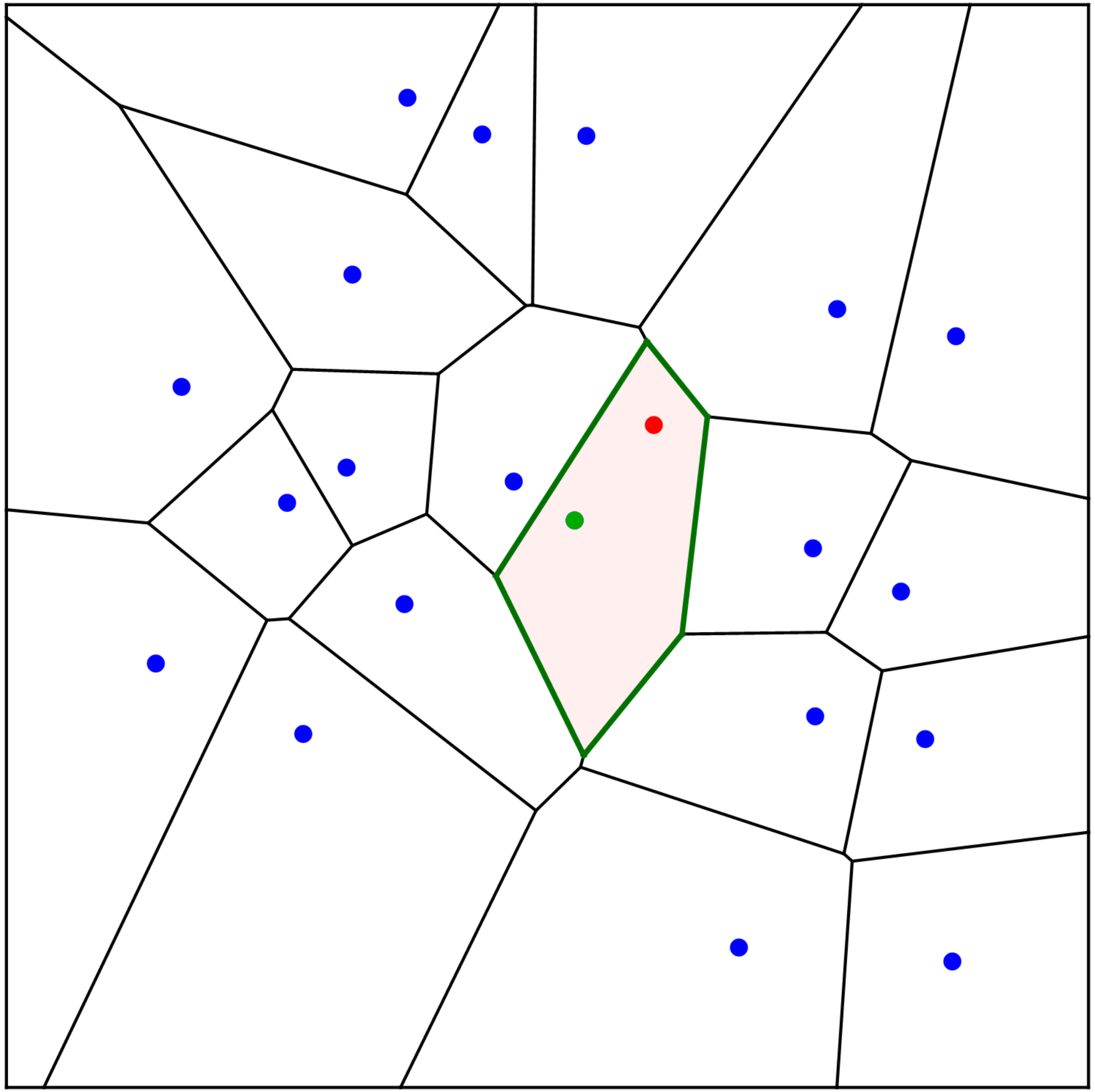

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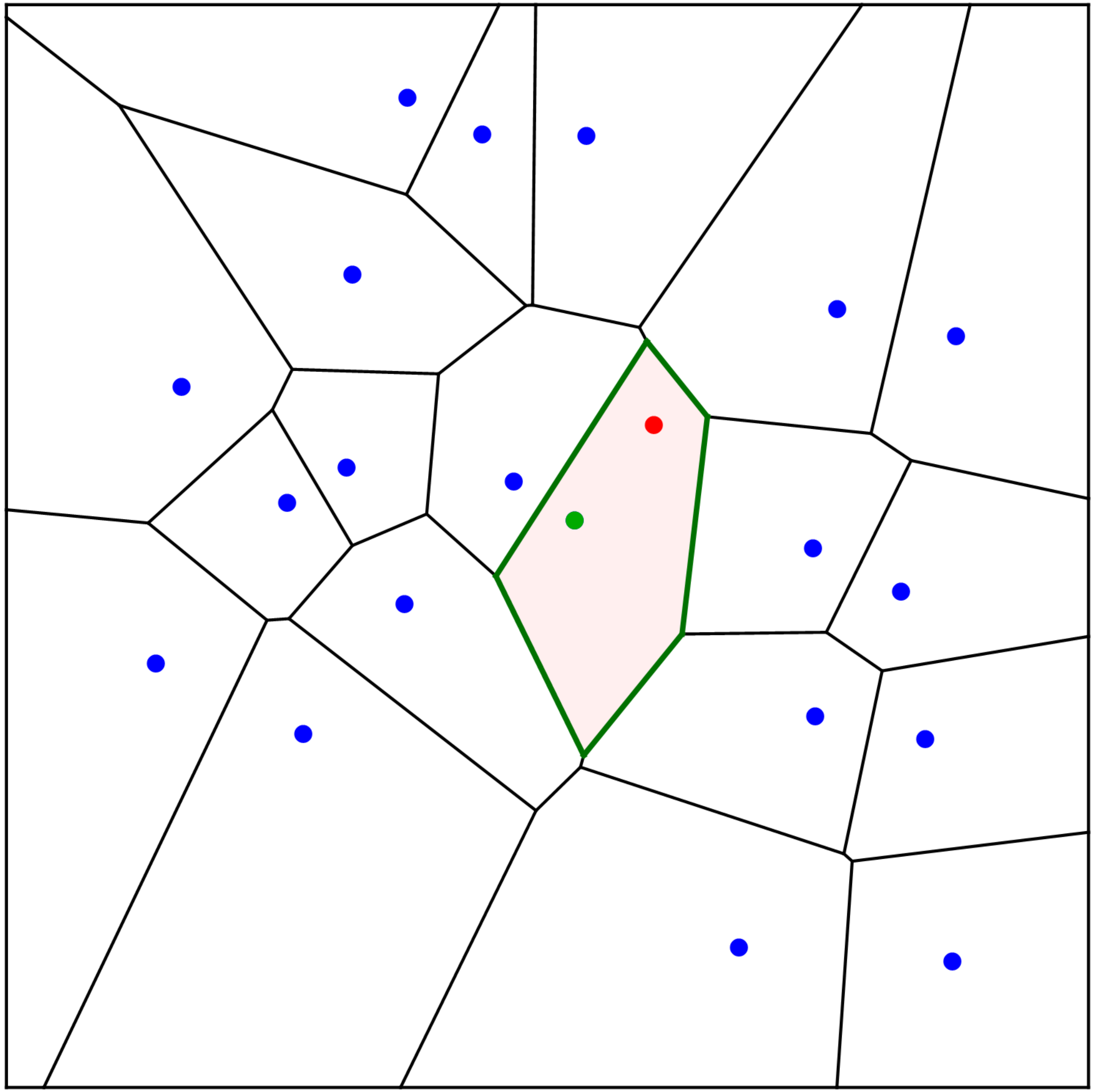

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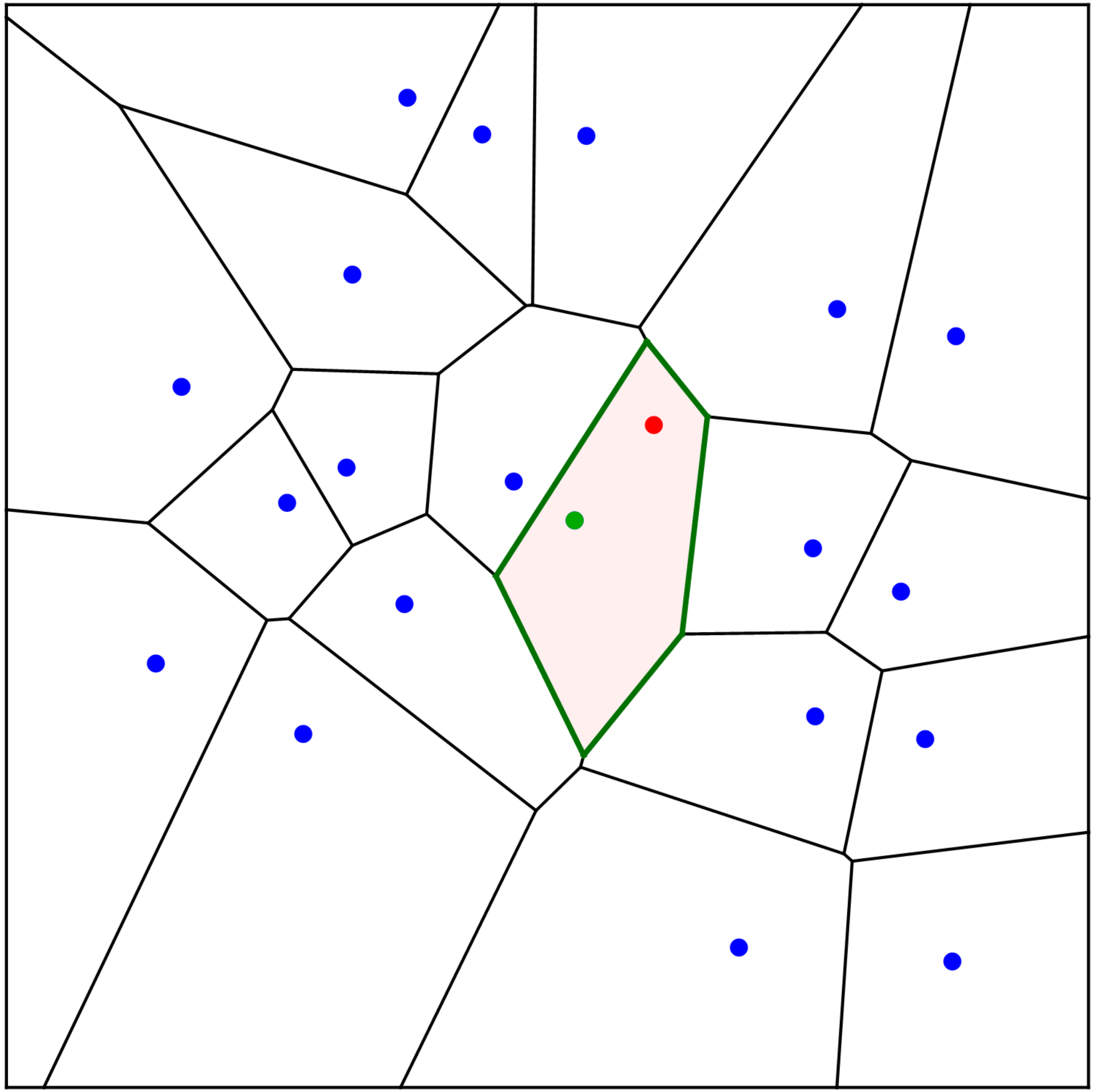
- Most of the applications are in high dimensions



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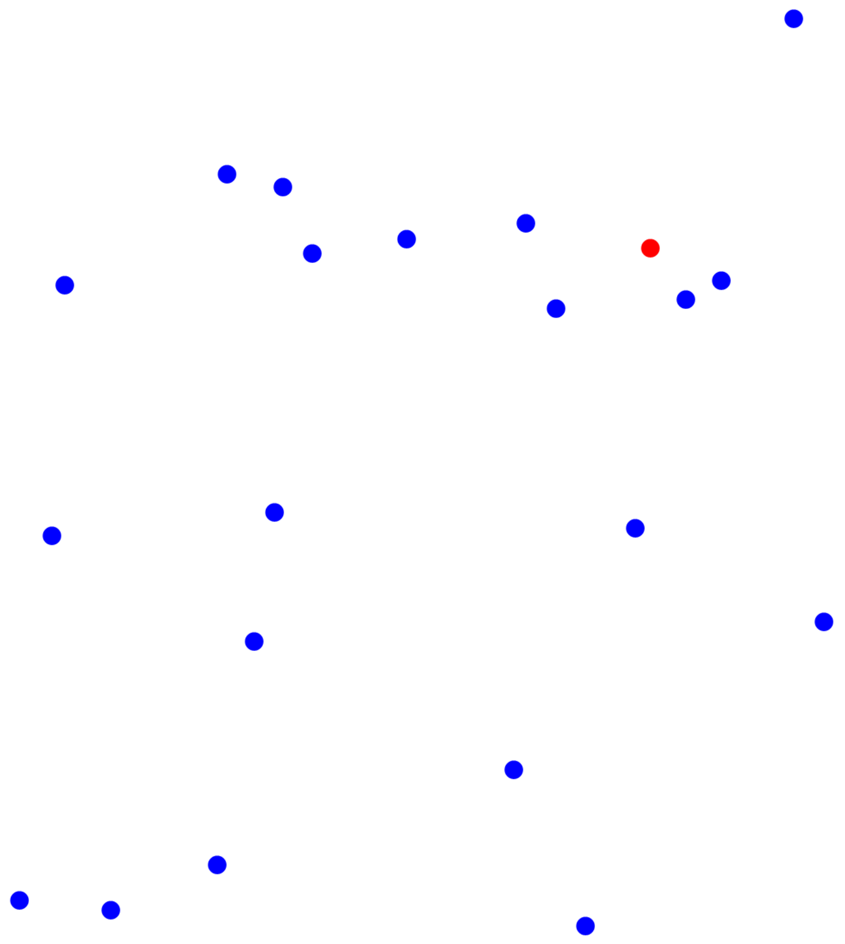

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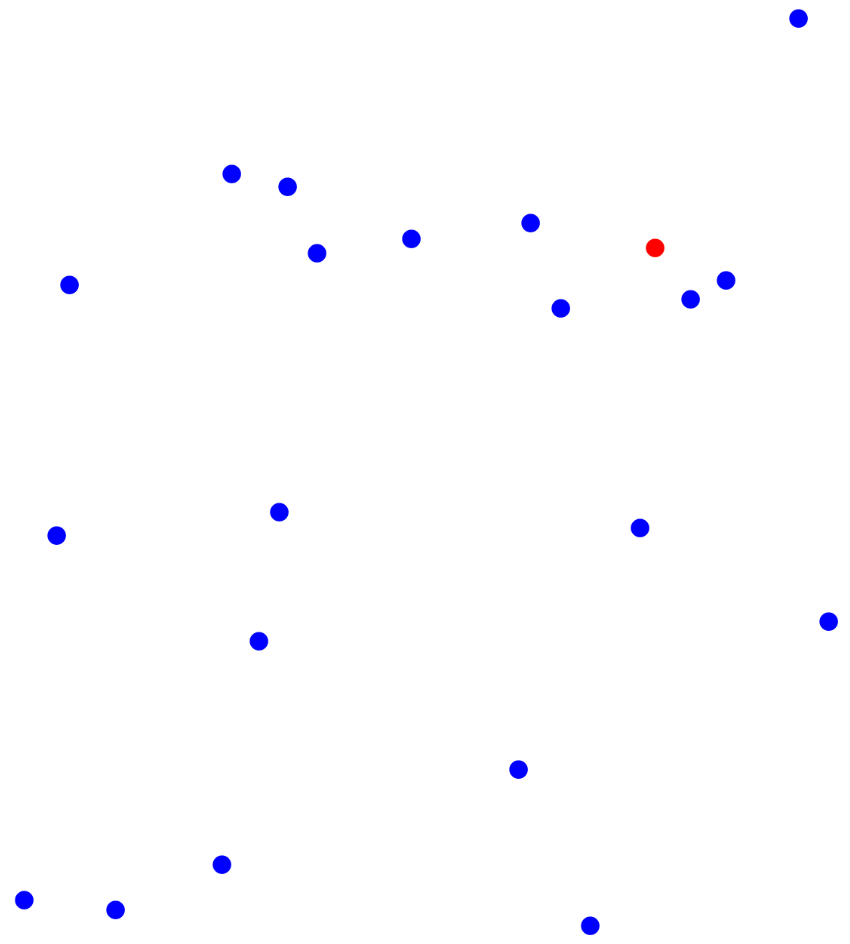

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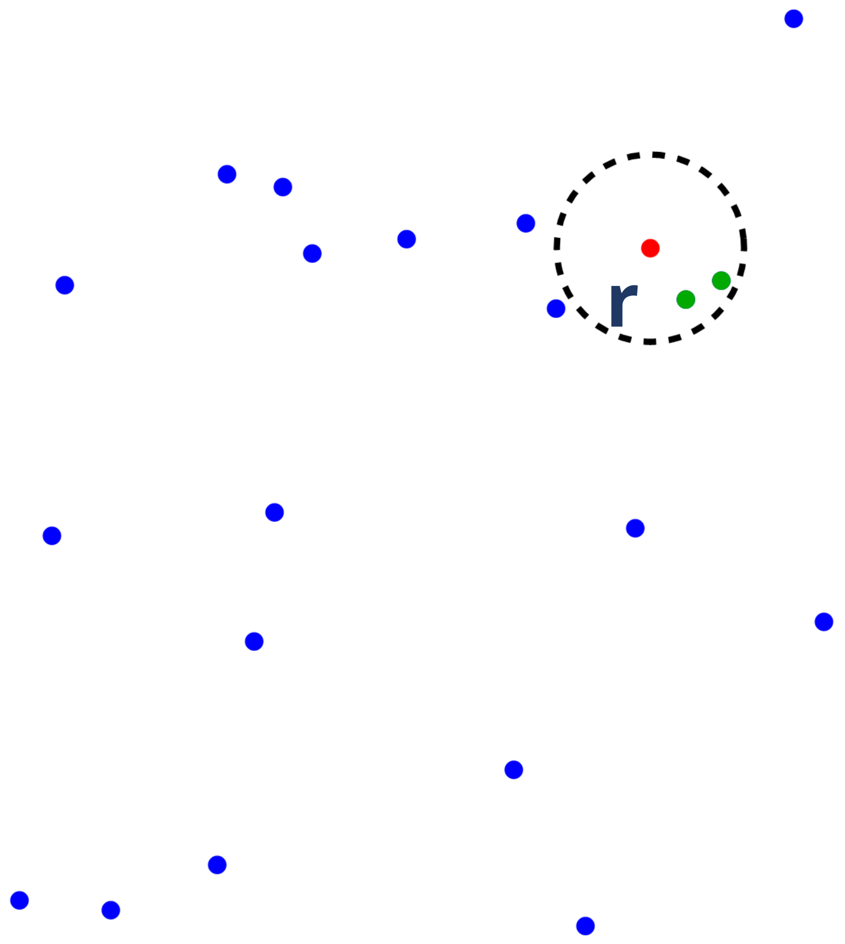

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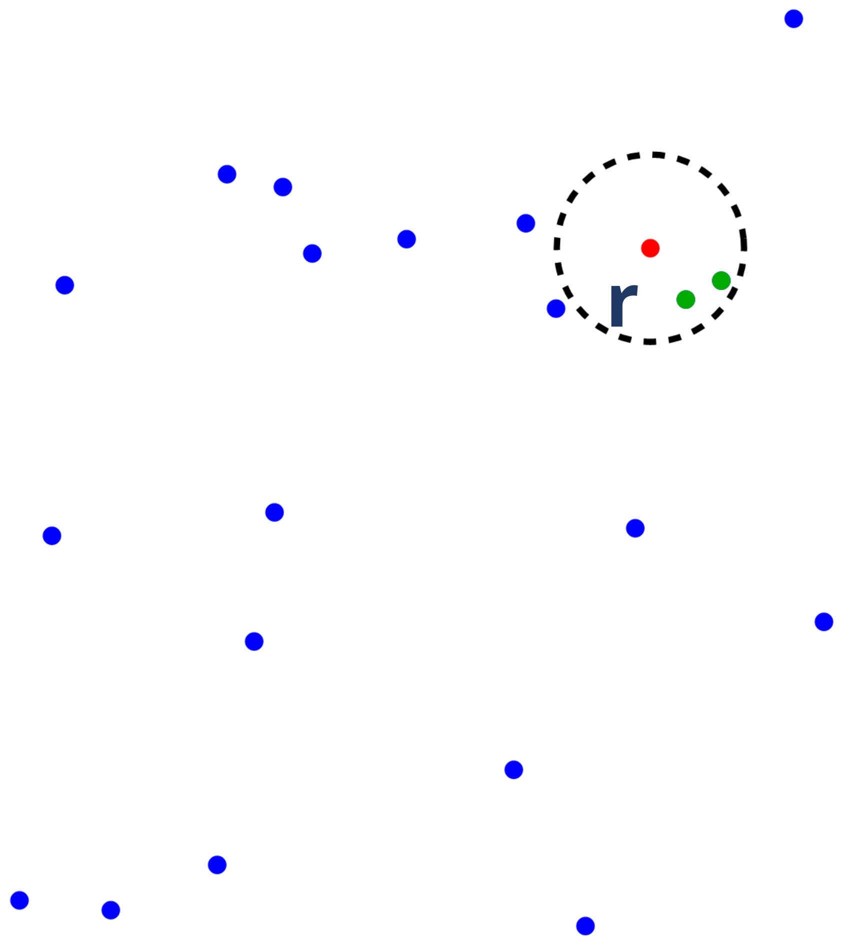

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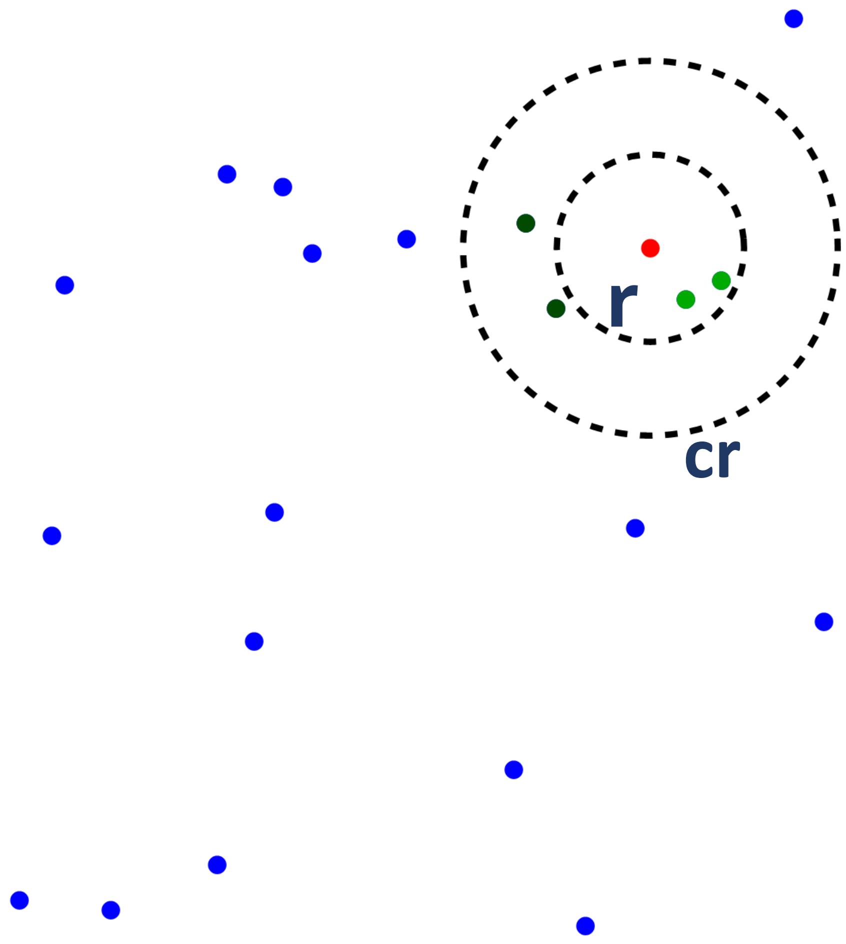

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- Cryptanalysis (the Shortest Vector Problem in lattices) **[Laarhoven 2015]**
- Optimization: Coordinate Descent **[Dhillon, Ravikumar, Tewari 2011]**, Stochastic Gradient Descent **[Hofmann, Lucchi, McWilliams 2015]** etc

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- Why interesting?
- **In theory:** can reduce general case to the spherical case (later in the talk)
- **In practice:**
  - Cosine similarity is widely used
  - Oftentimes, can boldly pretend that the dataset lies on a sphere and *be just fine*

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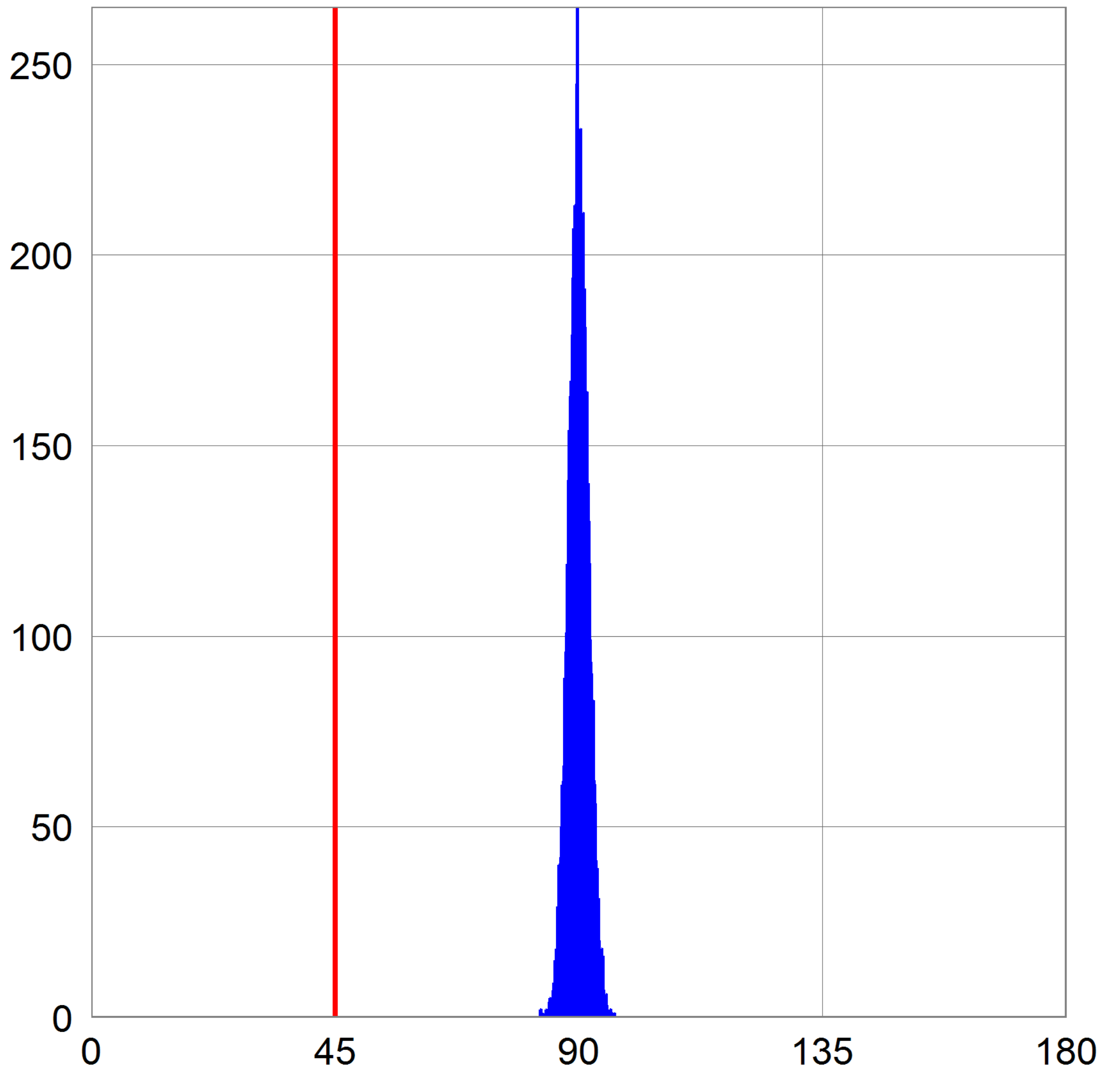

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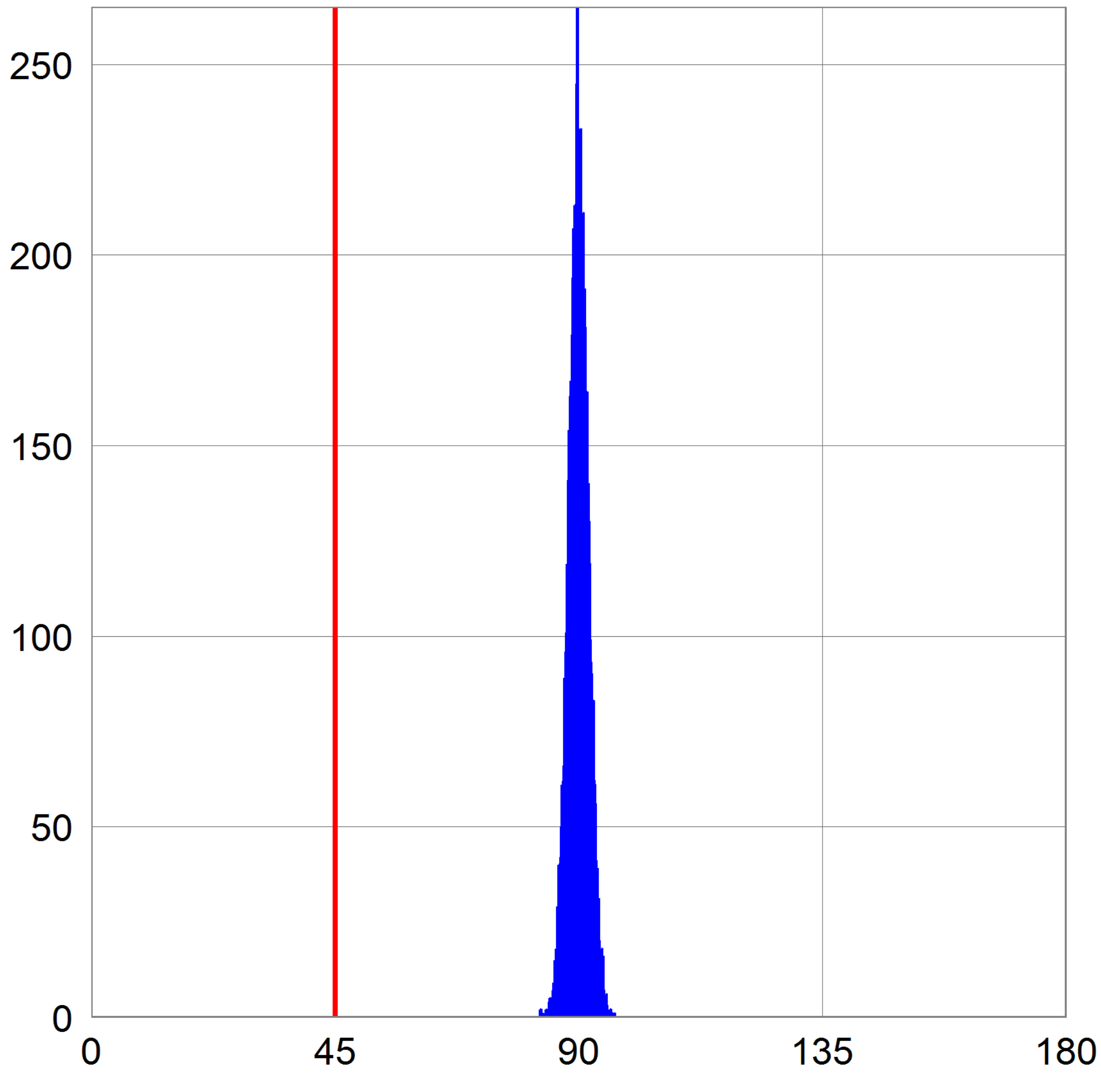
- Instructive case to think about
  - Concentration of angles around **90** degrees happens in practice



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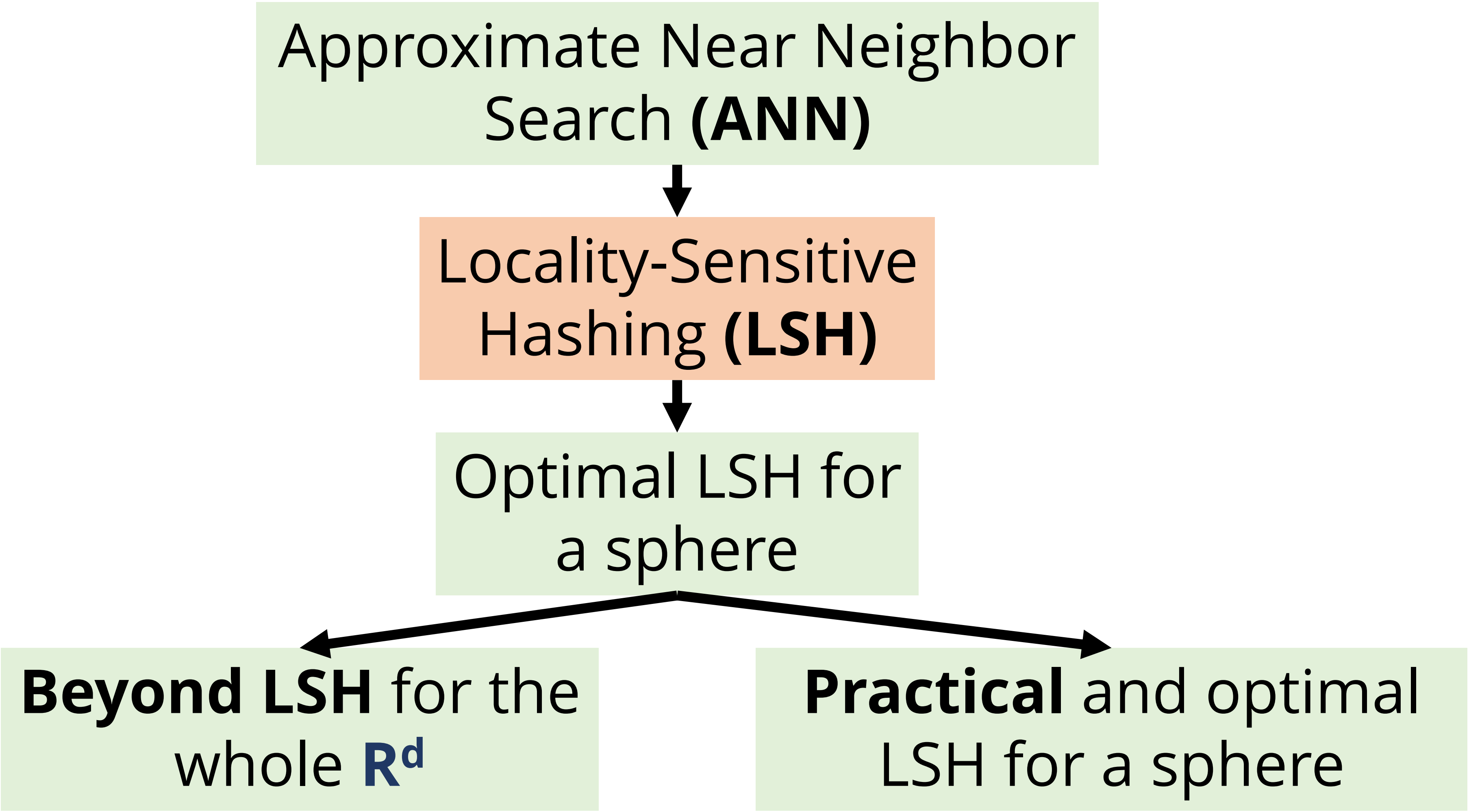
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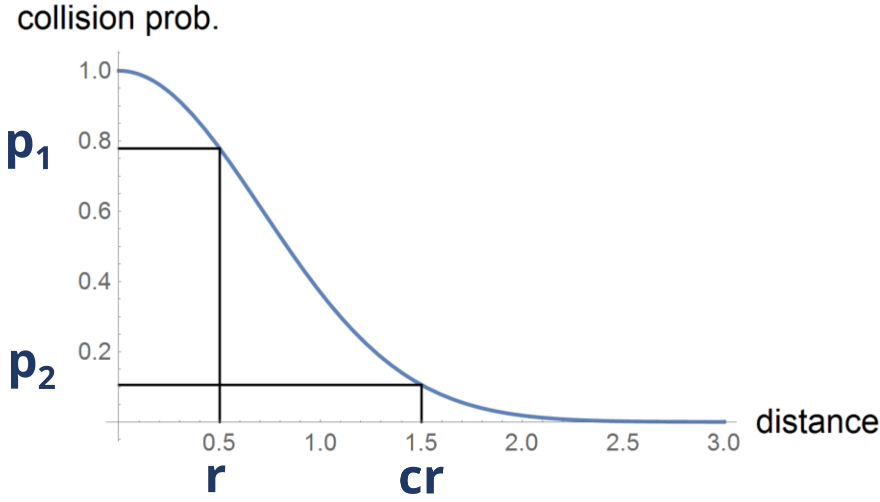

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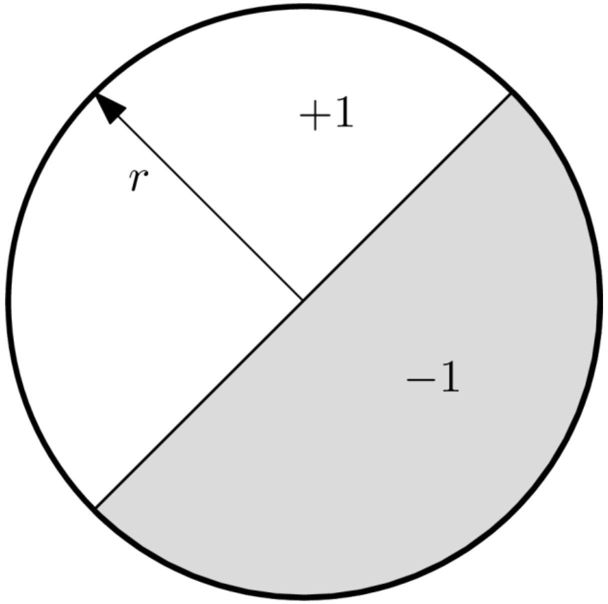

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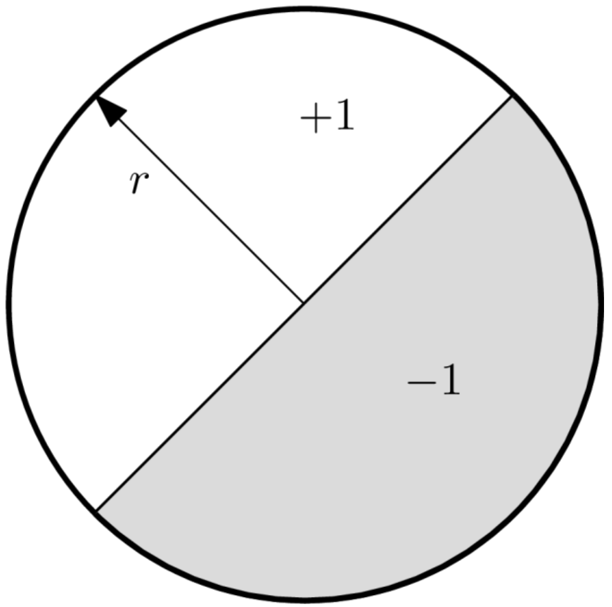

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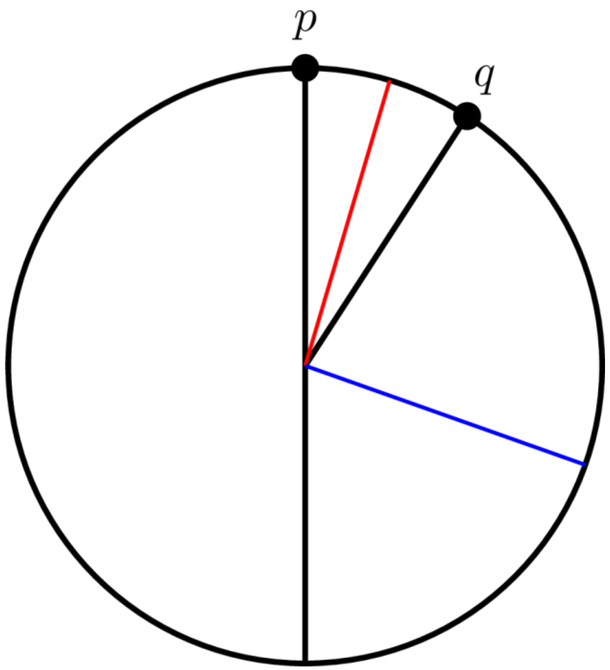

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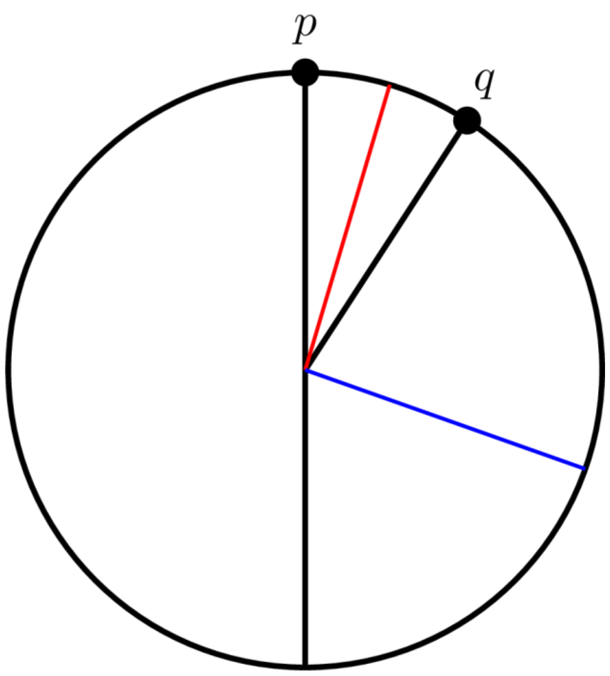

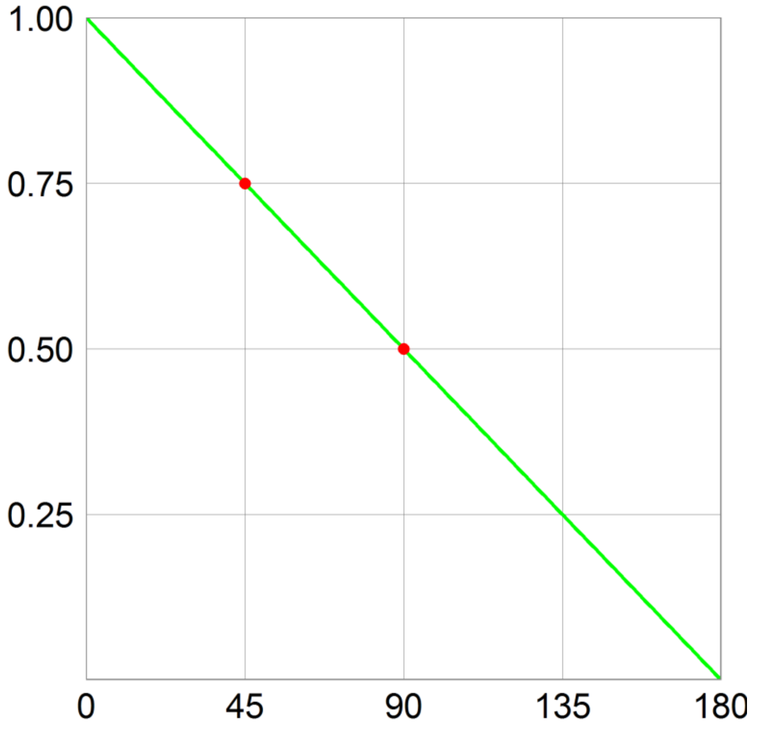

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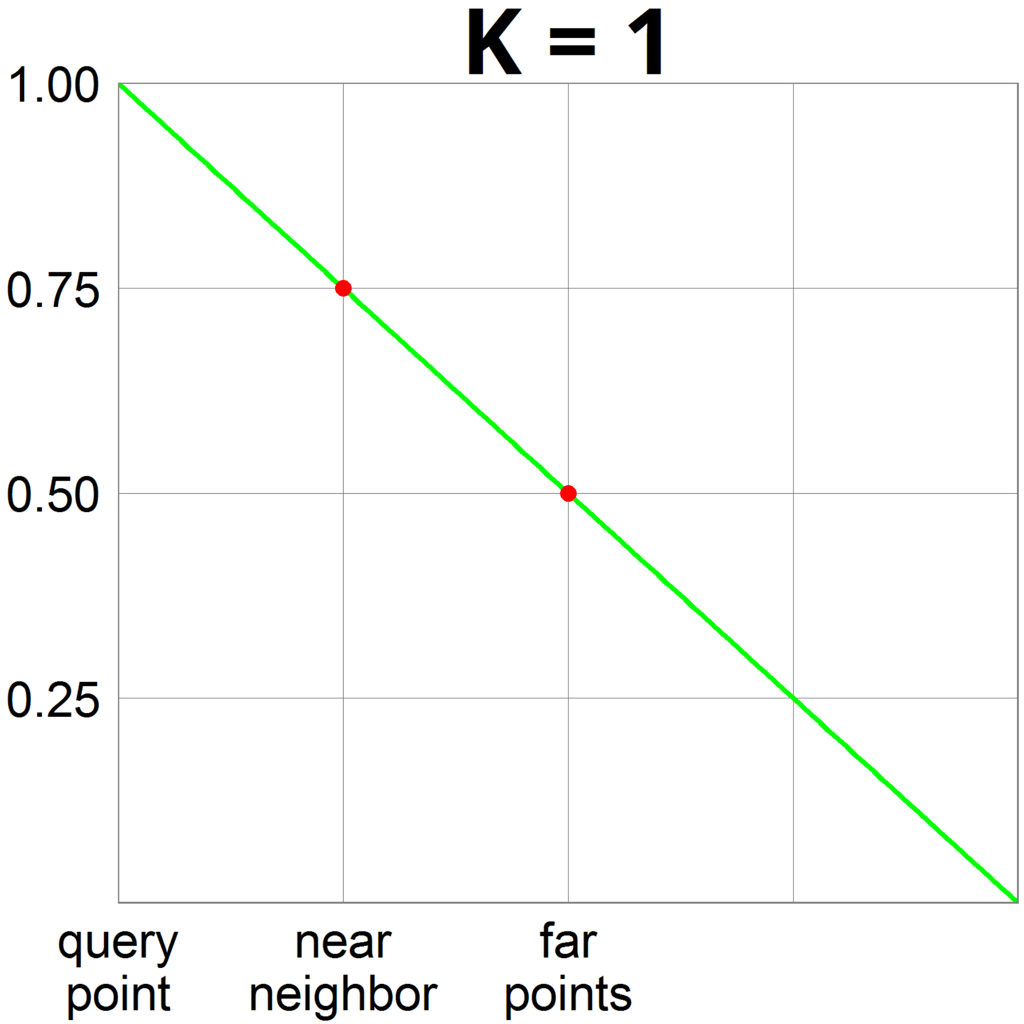

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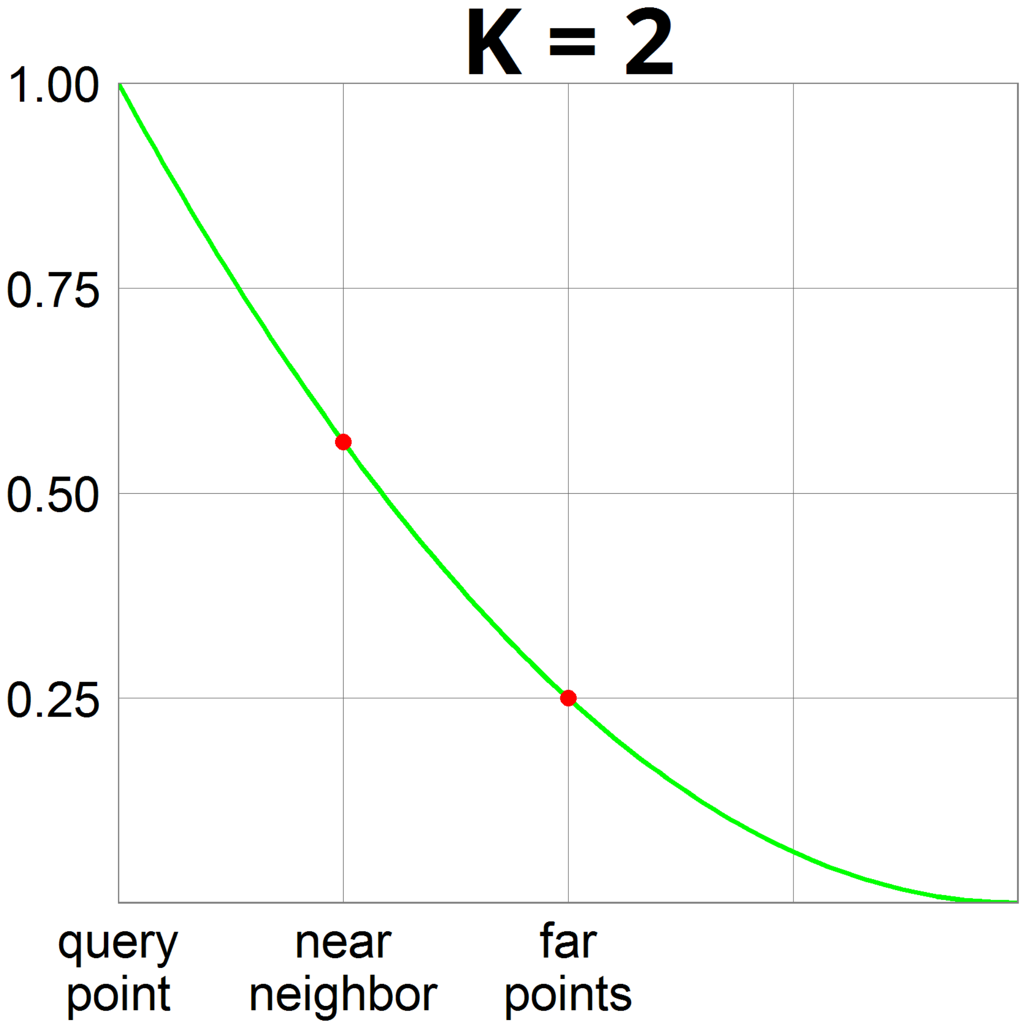

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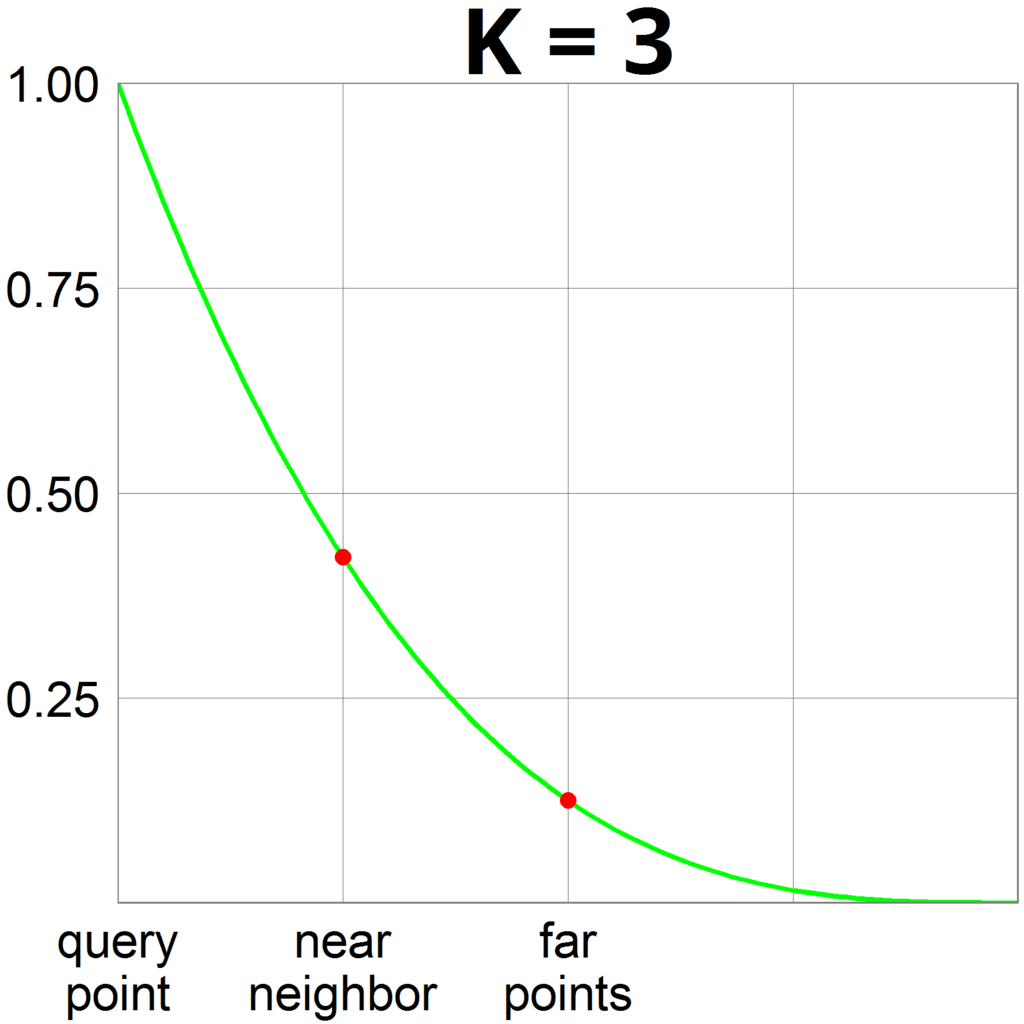

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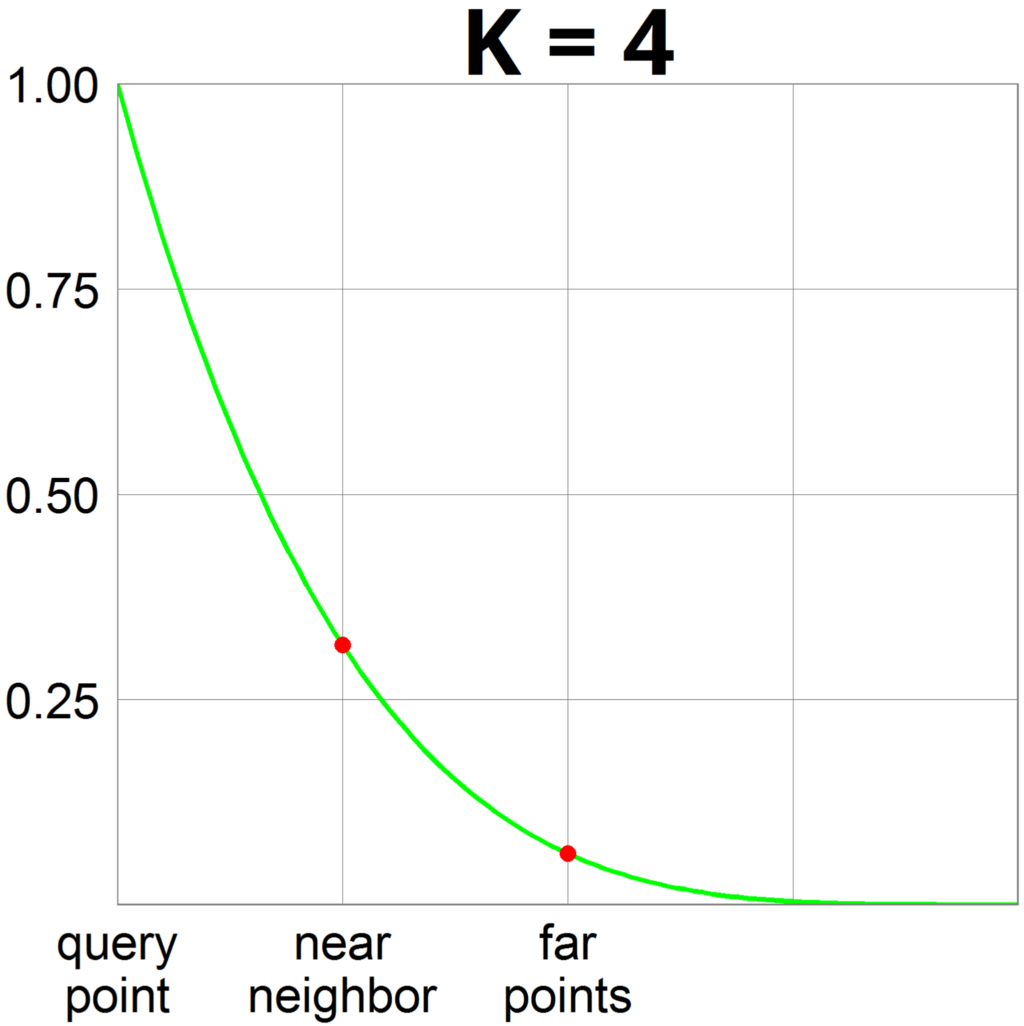

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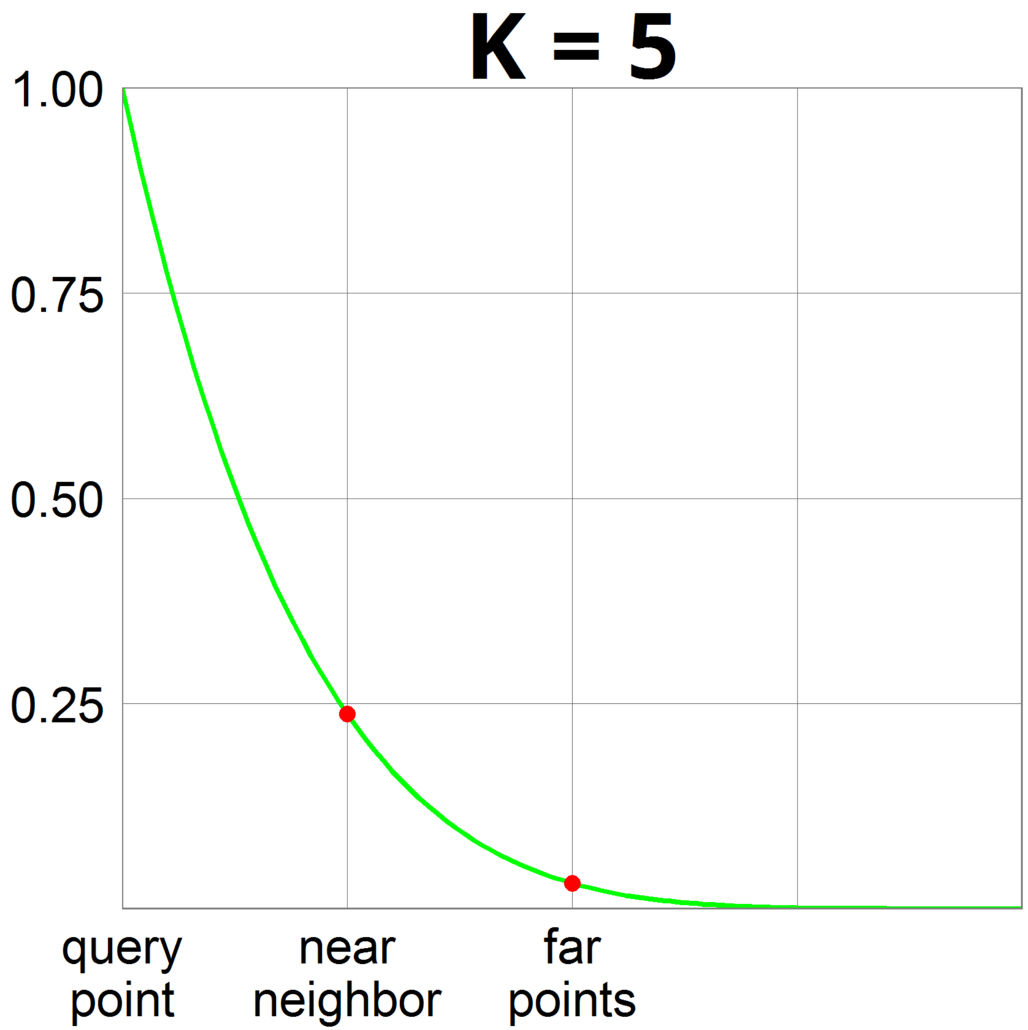

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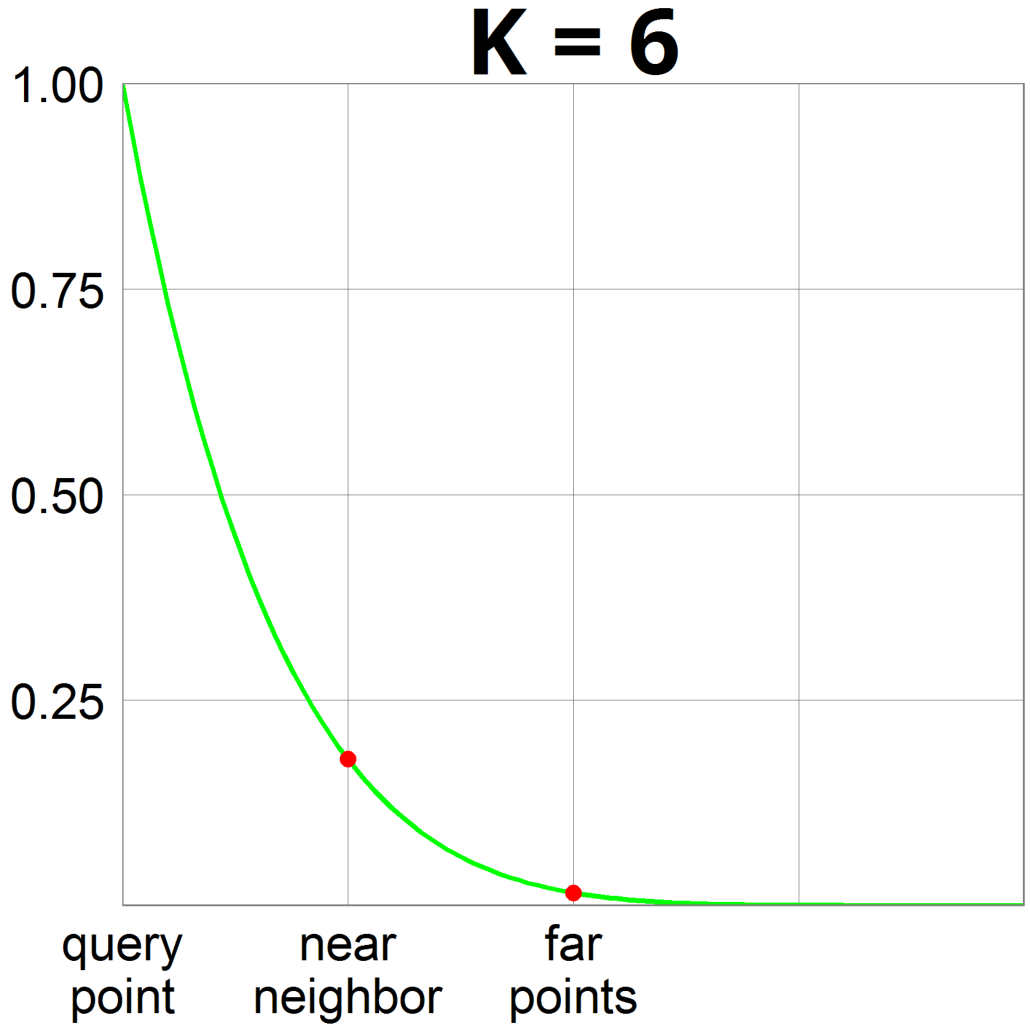

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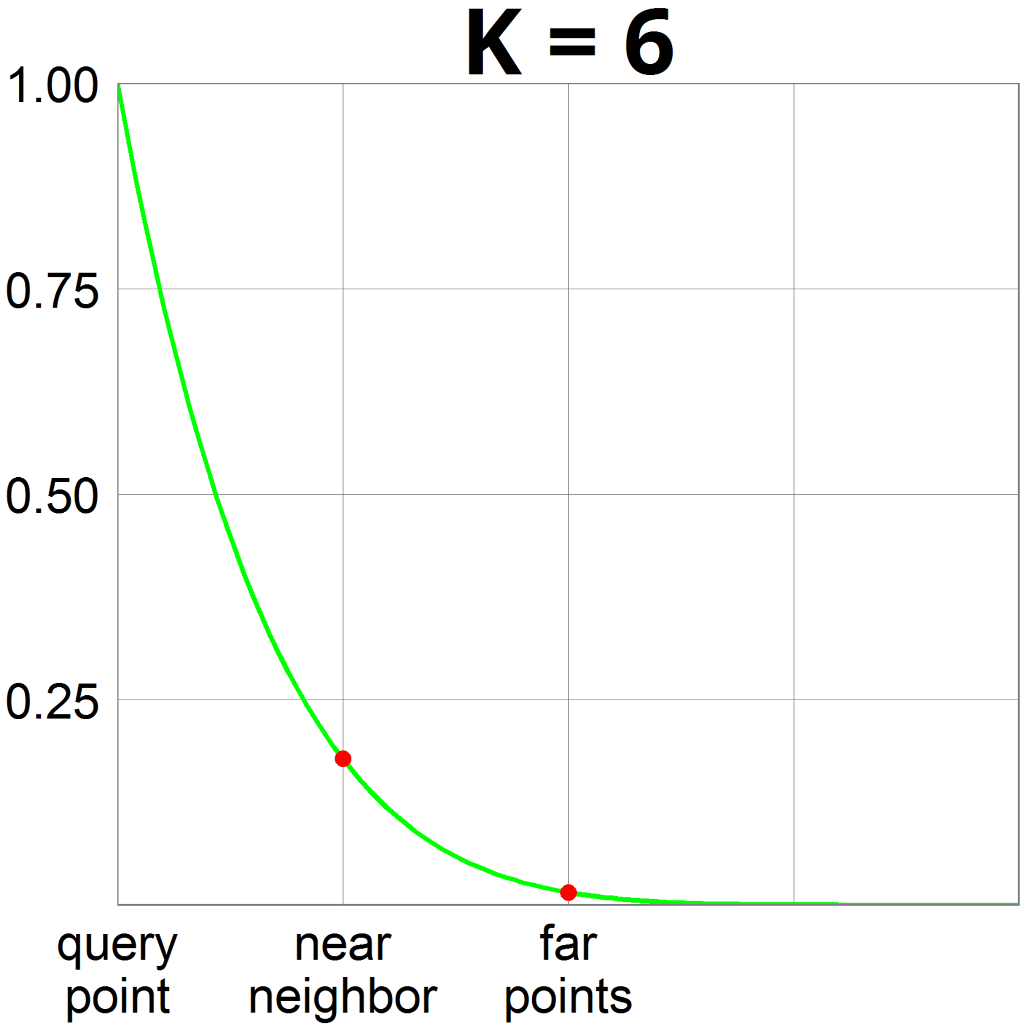

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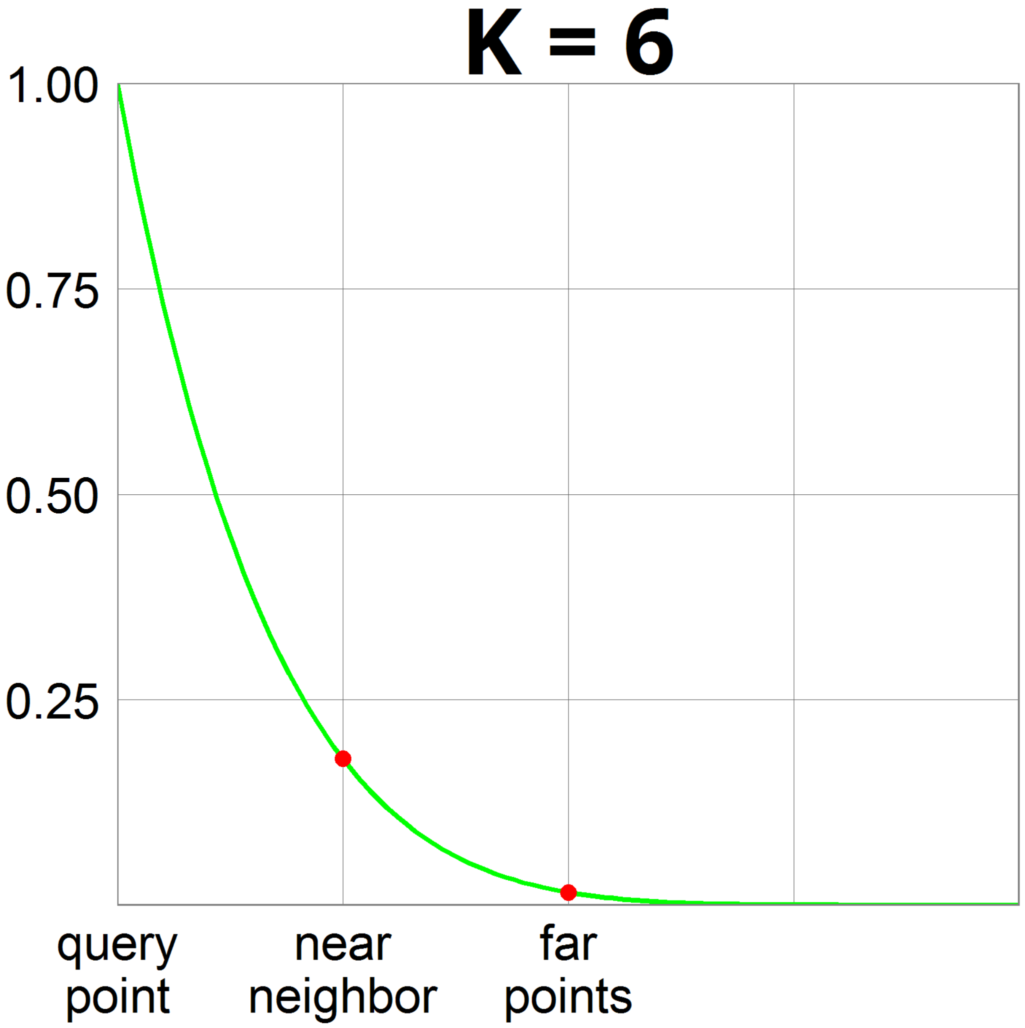
- Thus, need **L = O($n^{0.42}$)** tables to boost the success probability to **0.99**



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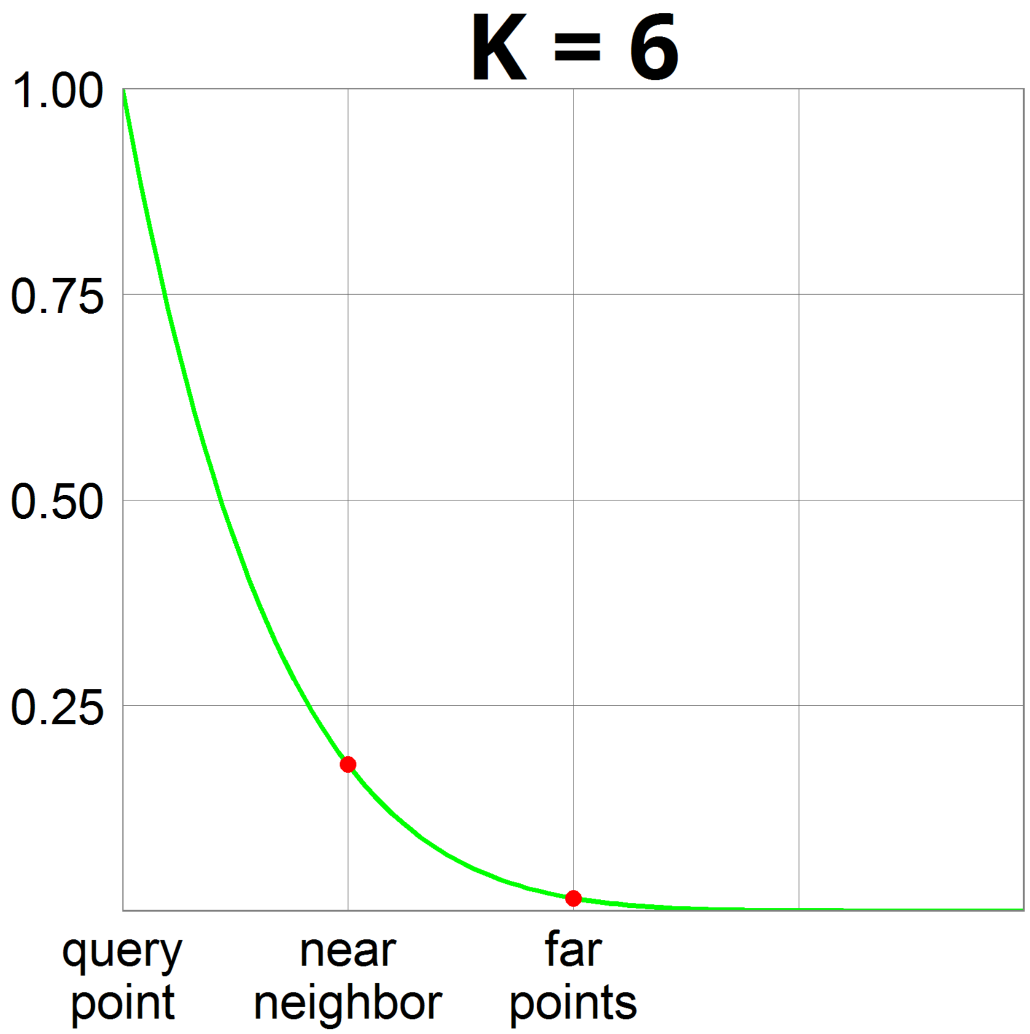
- Thus, need **L = $O(n^{0.42})$** tables to boost the success probability to **0.99**
- Overall: **$O(n^{1.42})$** space, **$O(n^{0.42})$** query time, **K·L** hyperplanes



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Recap:

- $\mathbf{p_1}$ is collision probability for close pairs
- $\mathbf{p_2}$ — for far pairs

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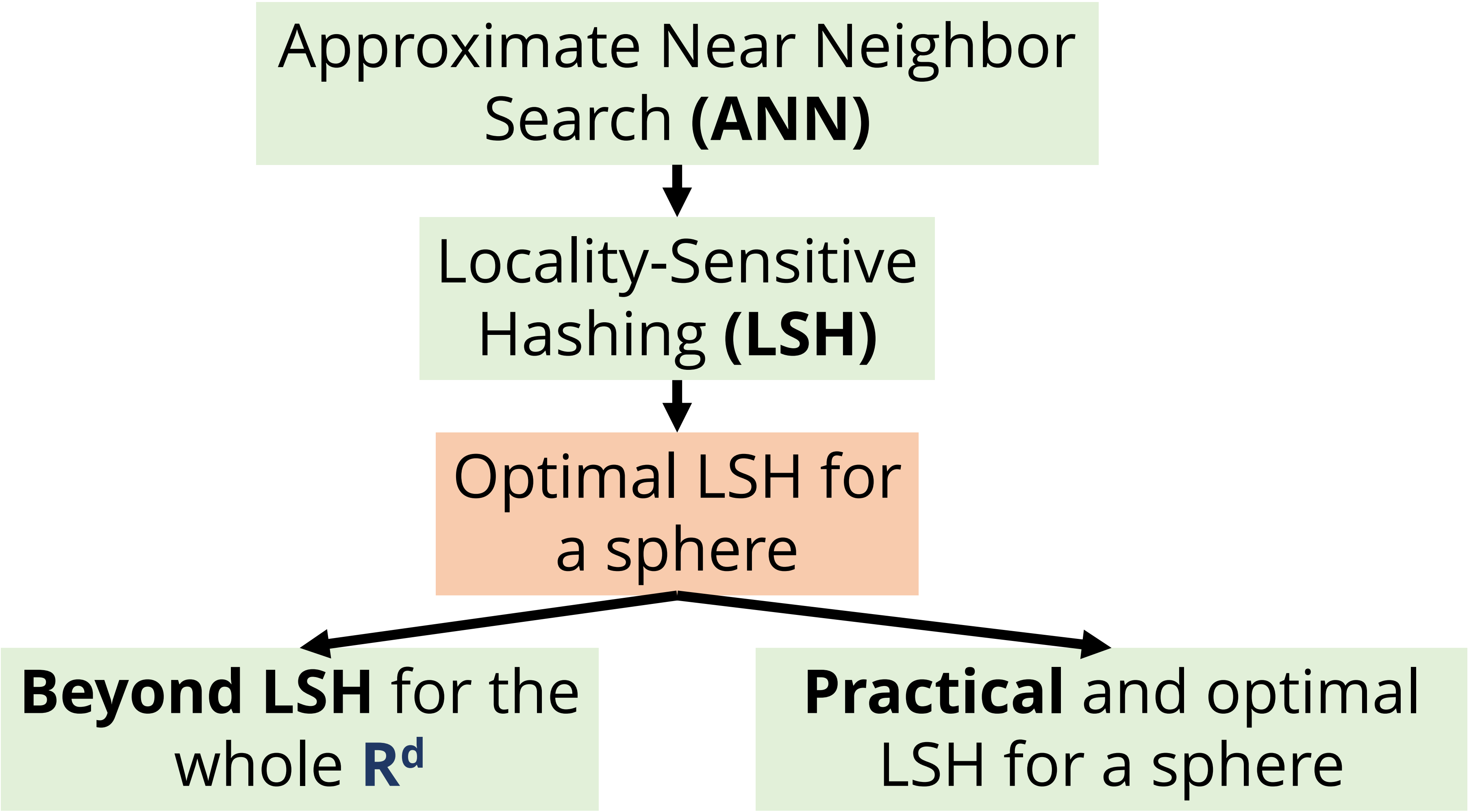
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  - Works for the general case of ANN on a sphere!

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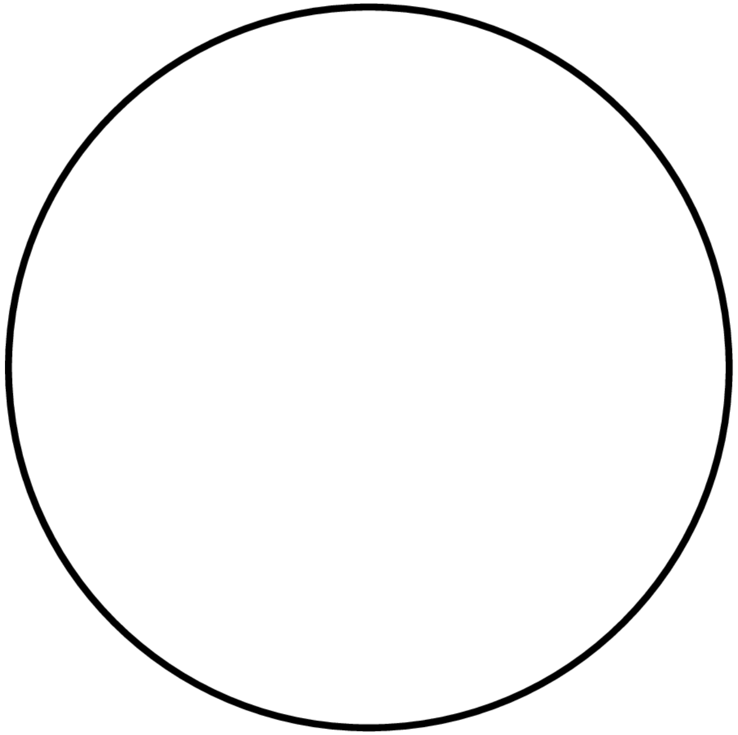

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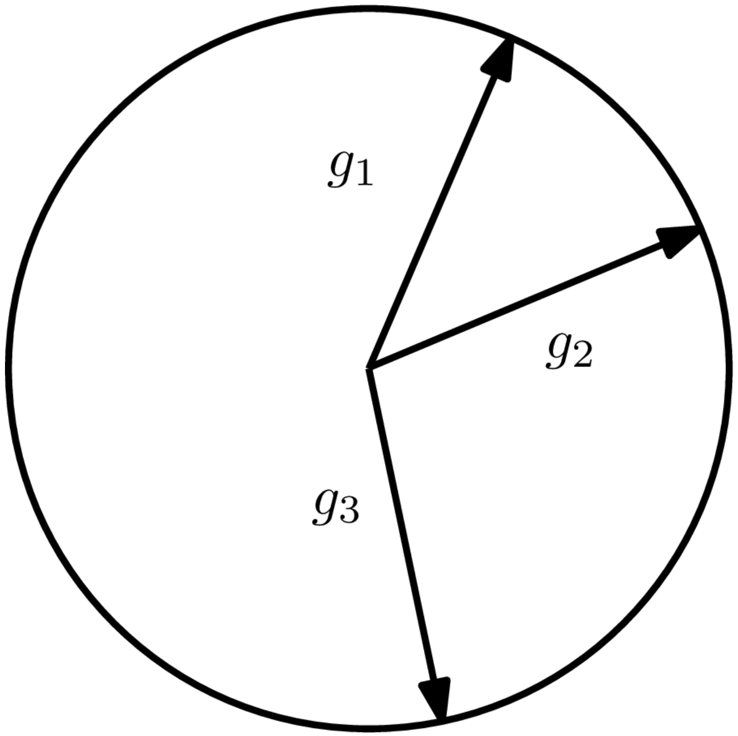

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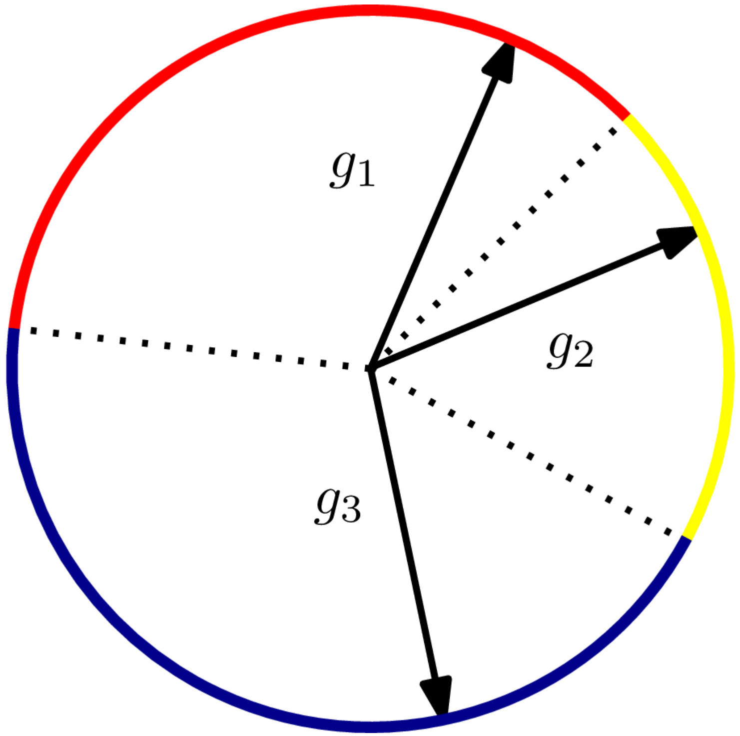

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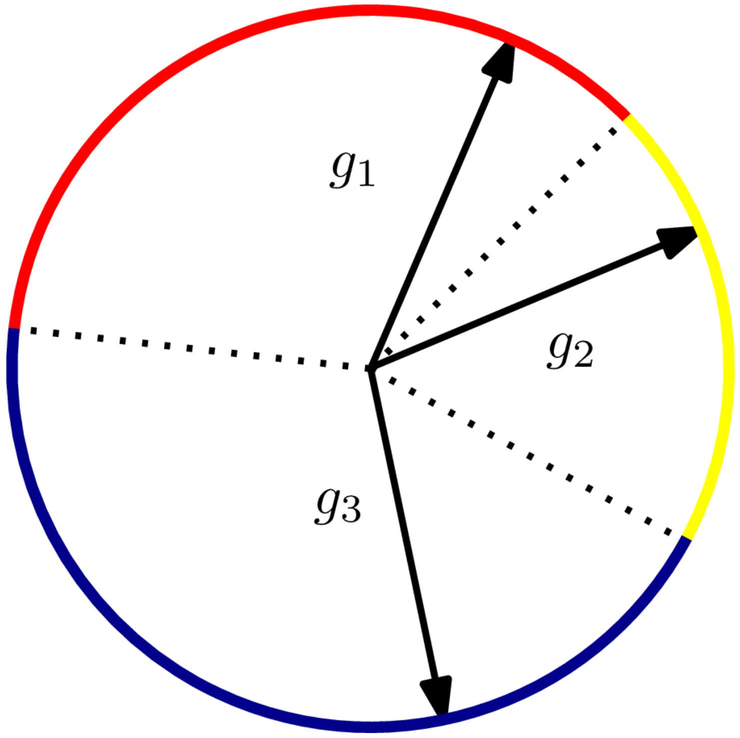
- $\mathbf{T = 2}$ is simply Hyperplane LSH



Hyperplane LSH vs. Voronoi LSH

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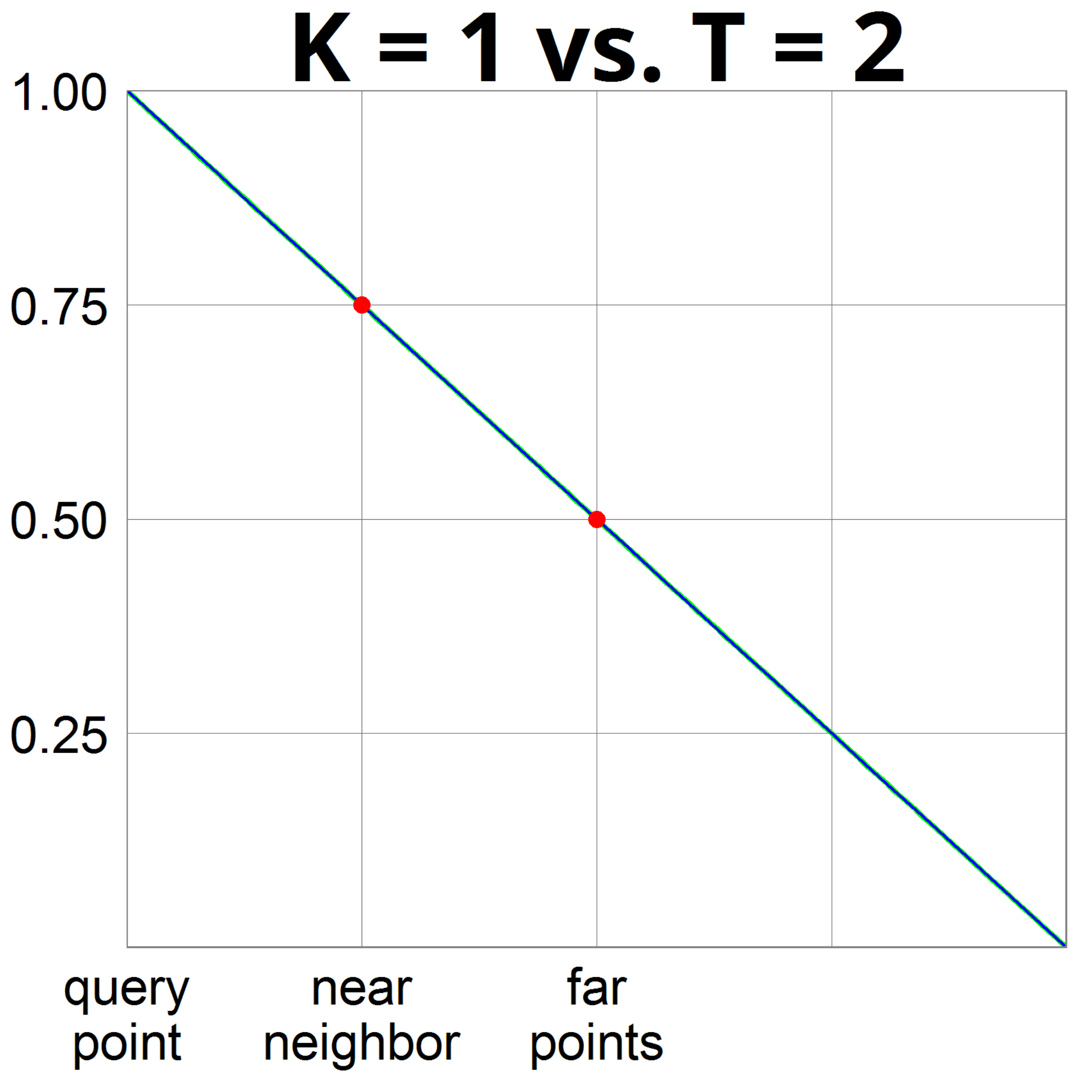

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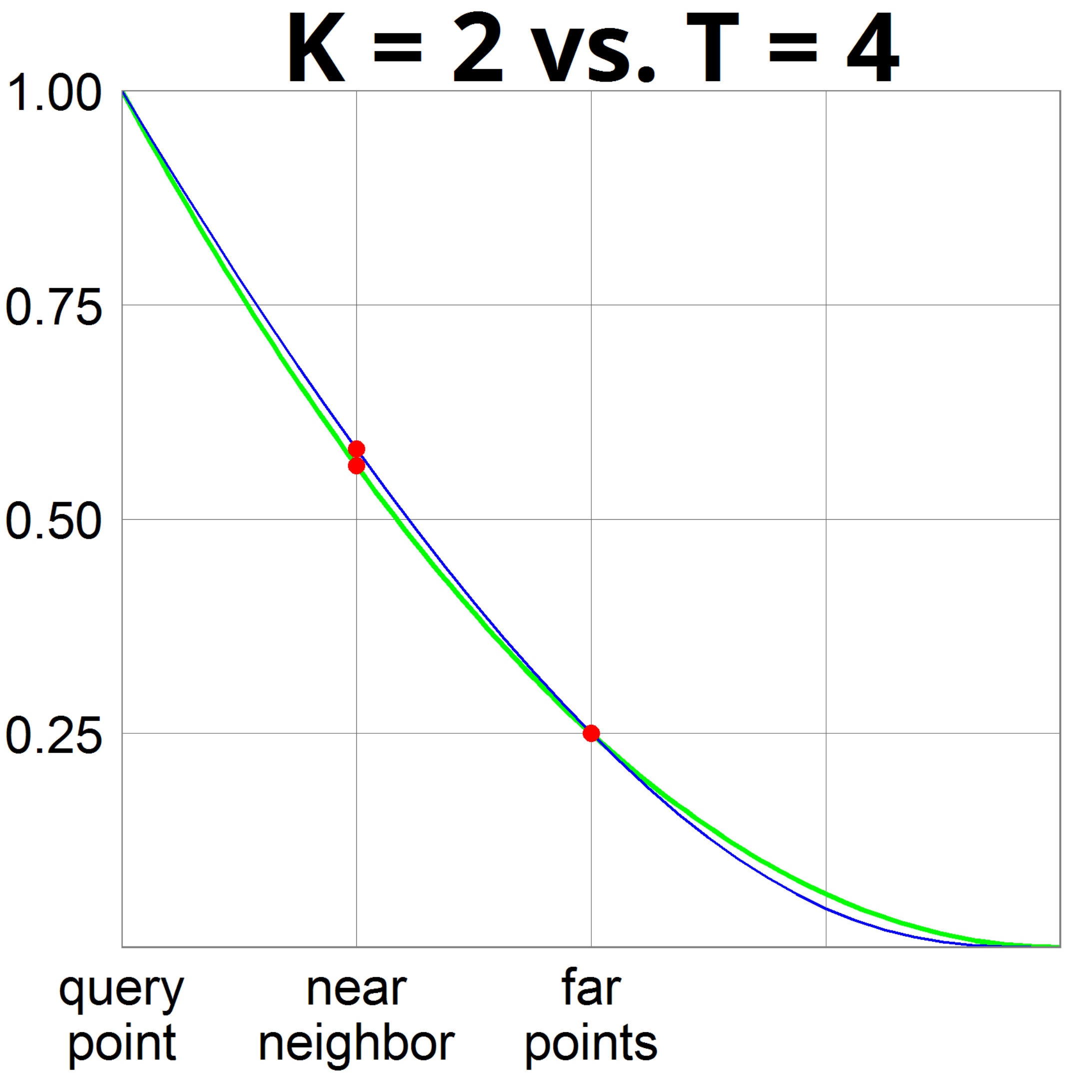

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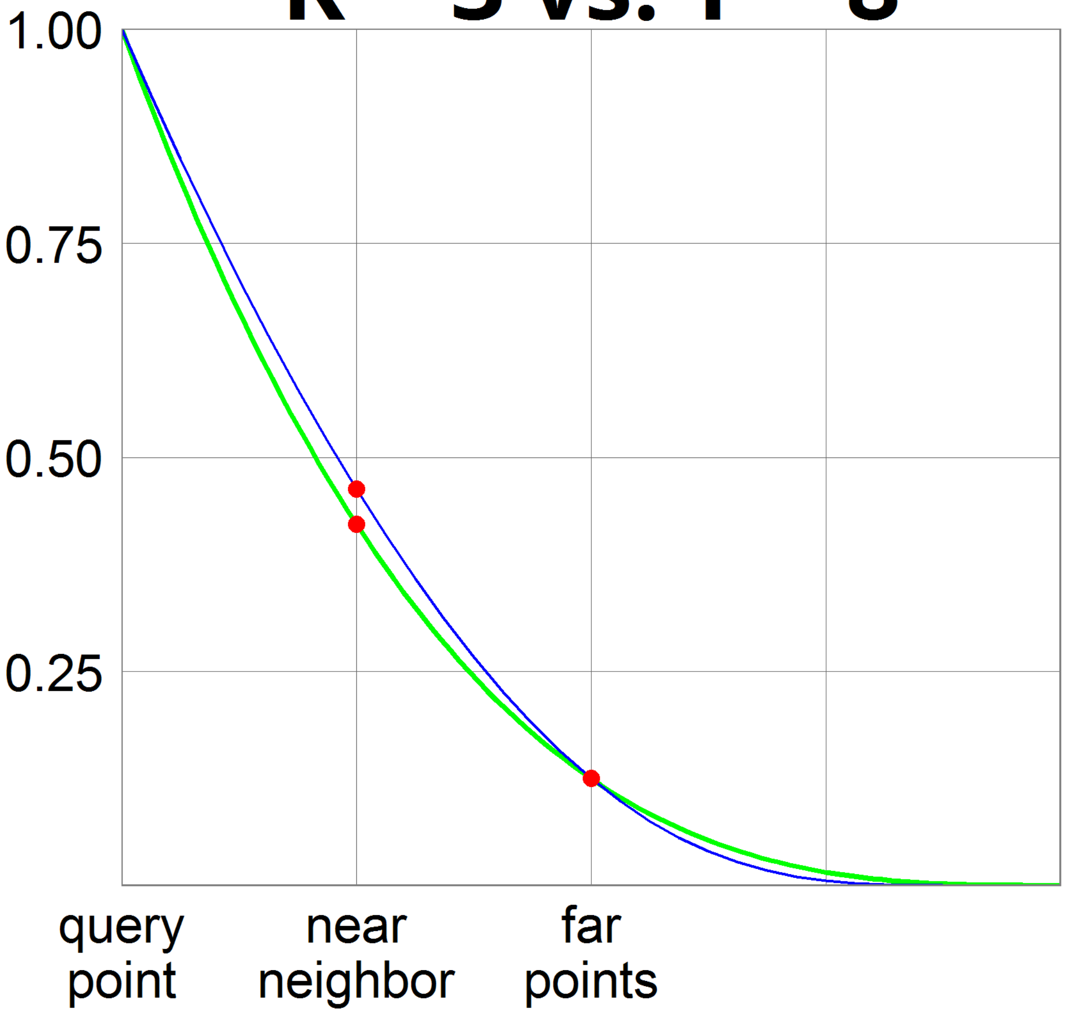
K = 3 vs. T = 8



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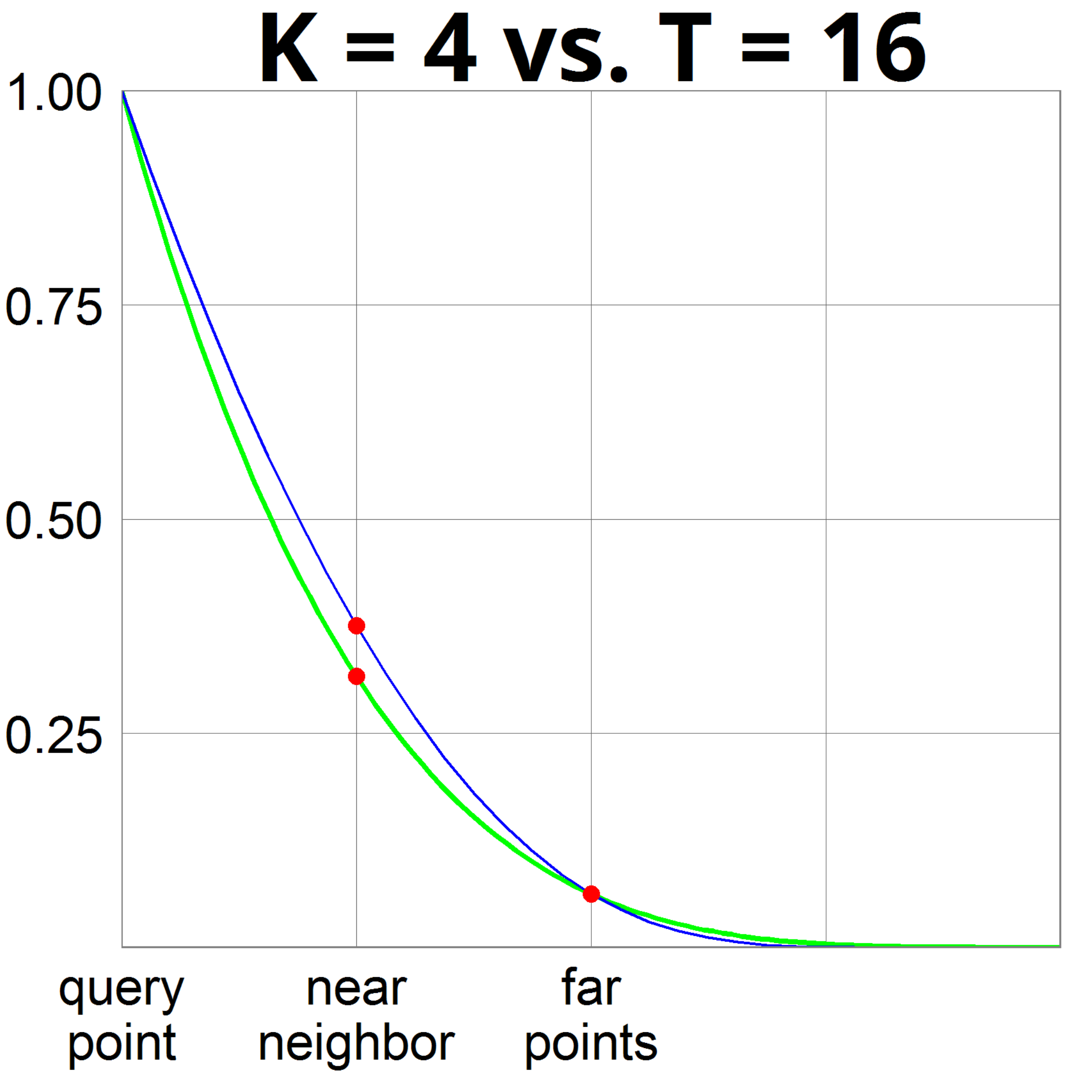

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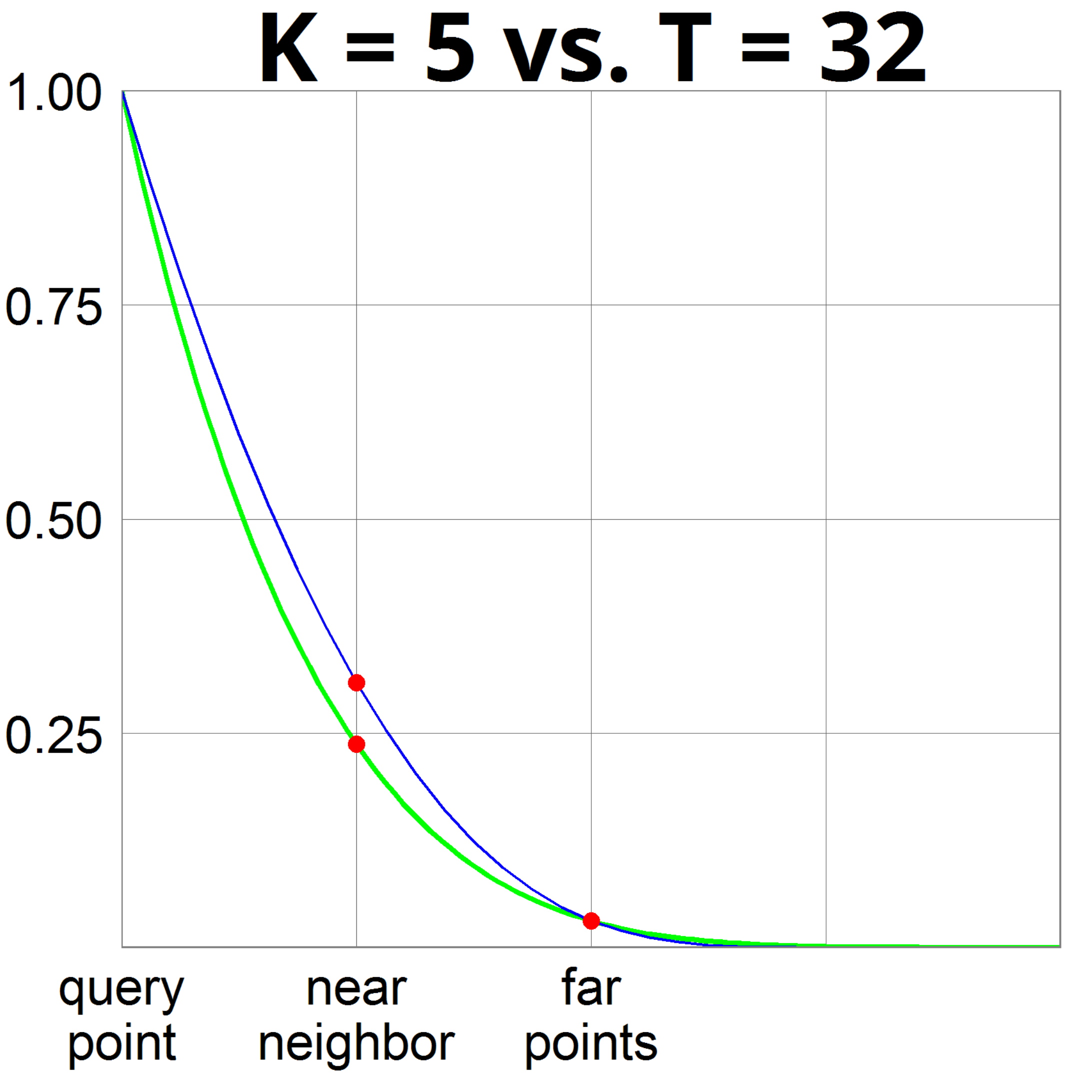

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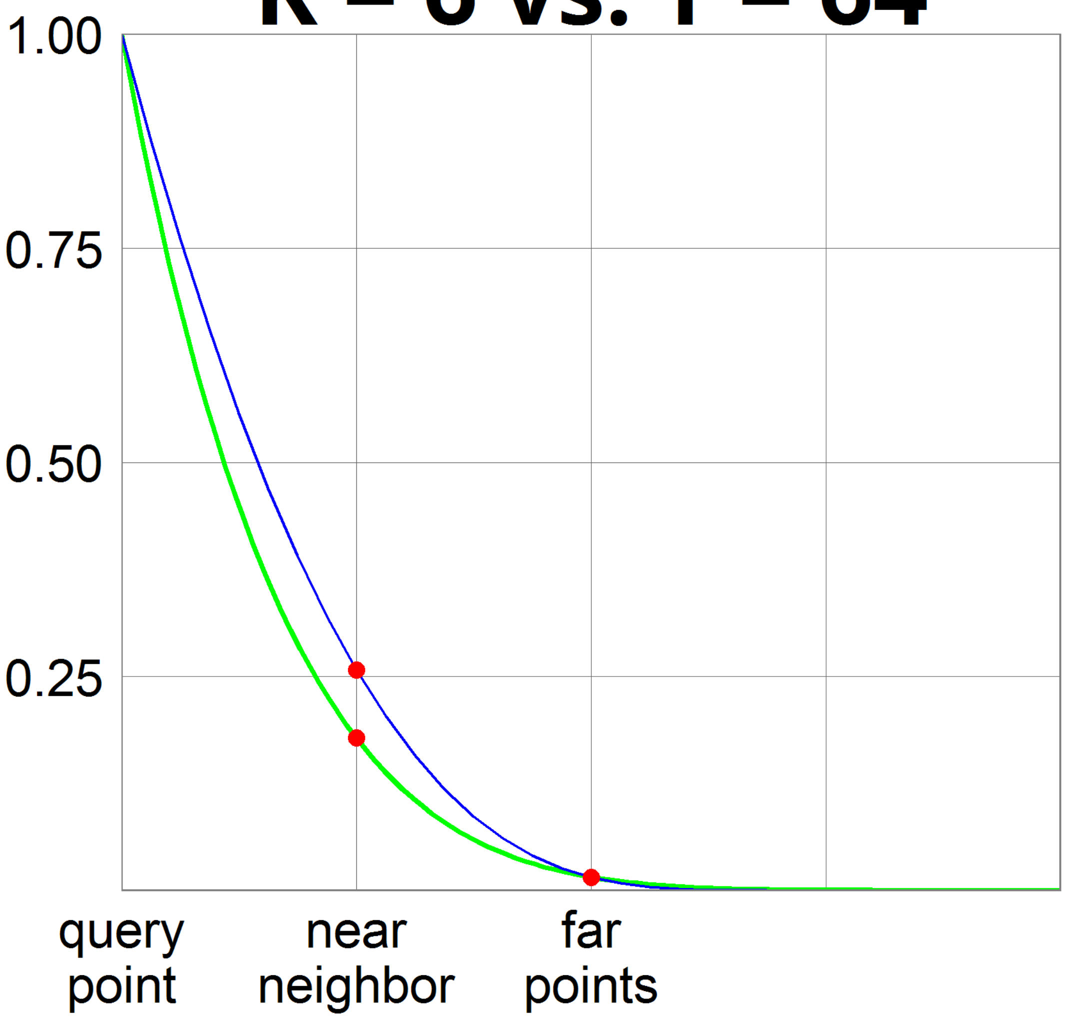

Hyperplane LSH vs. Voronoi LSH

- Let us compare **K** hyperplanes vs. Voronoi LSH with **$T = 2^K$** (in both cases **K**-bit hashes)
- As **T** grows, the gap between Hyperplane LSH and Voronoi LSH increases and

$$\rho = \ln(1/p_1) / \ln(1/p_2)$$

approaches **0.18**

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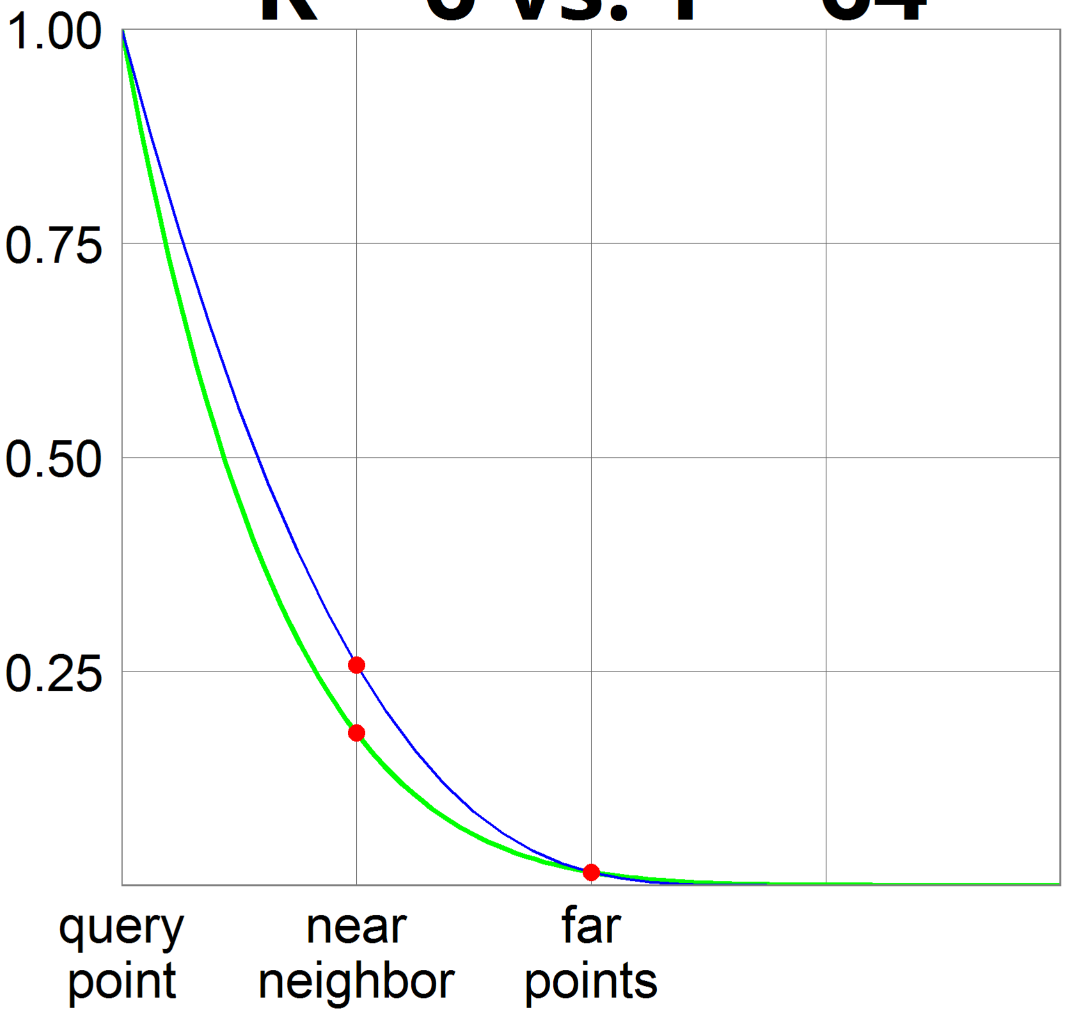

Outline

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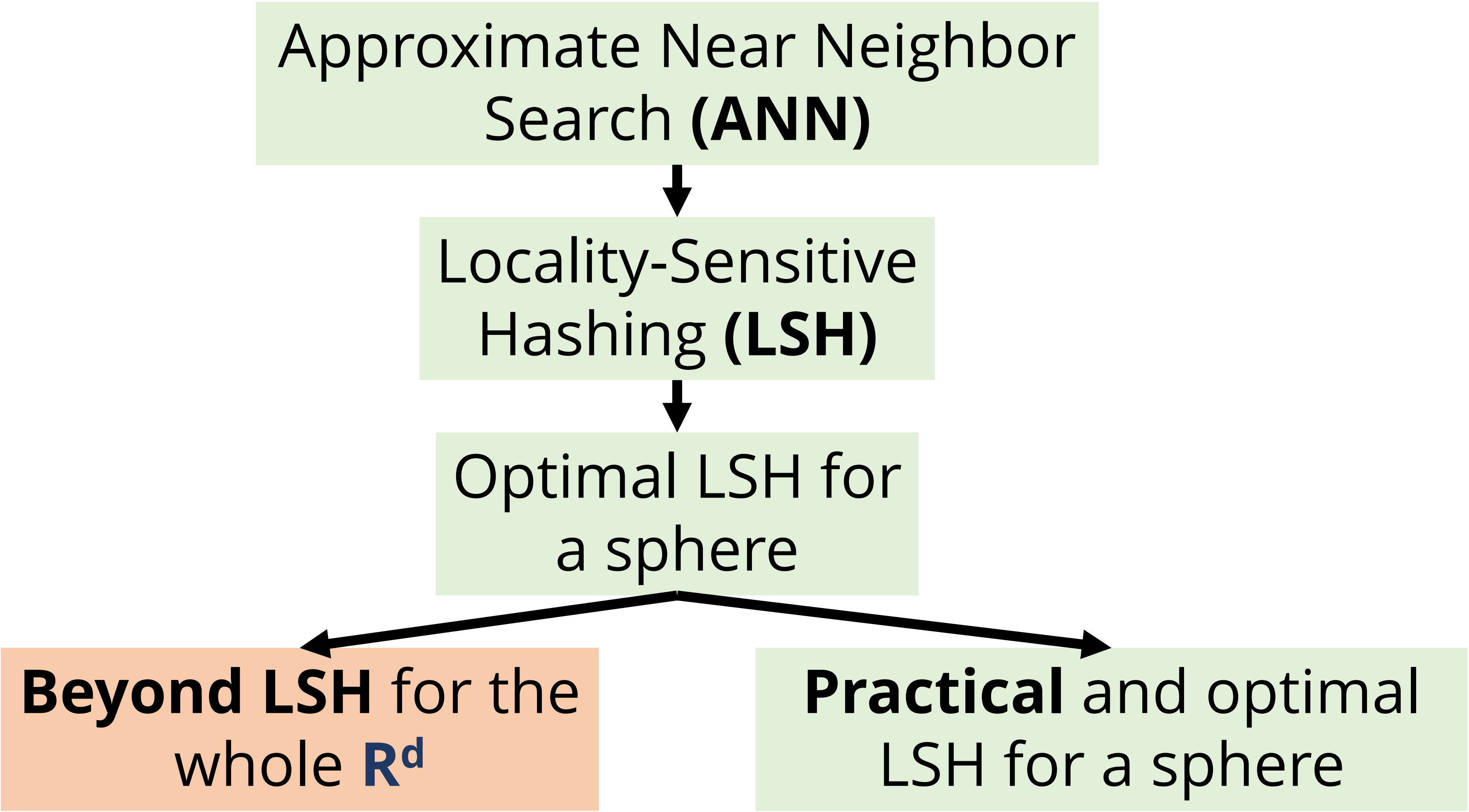
Approximate Near Neighbor Search **(ANN)**
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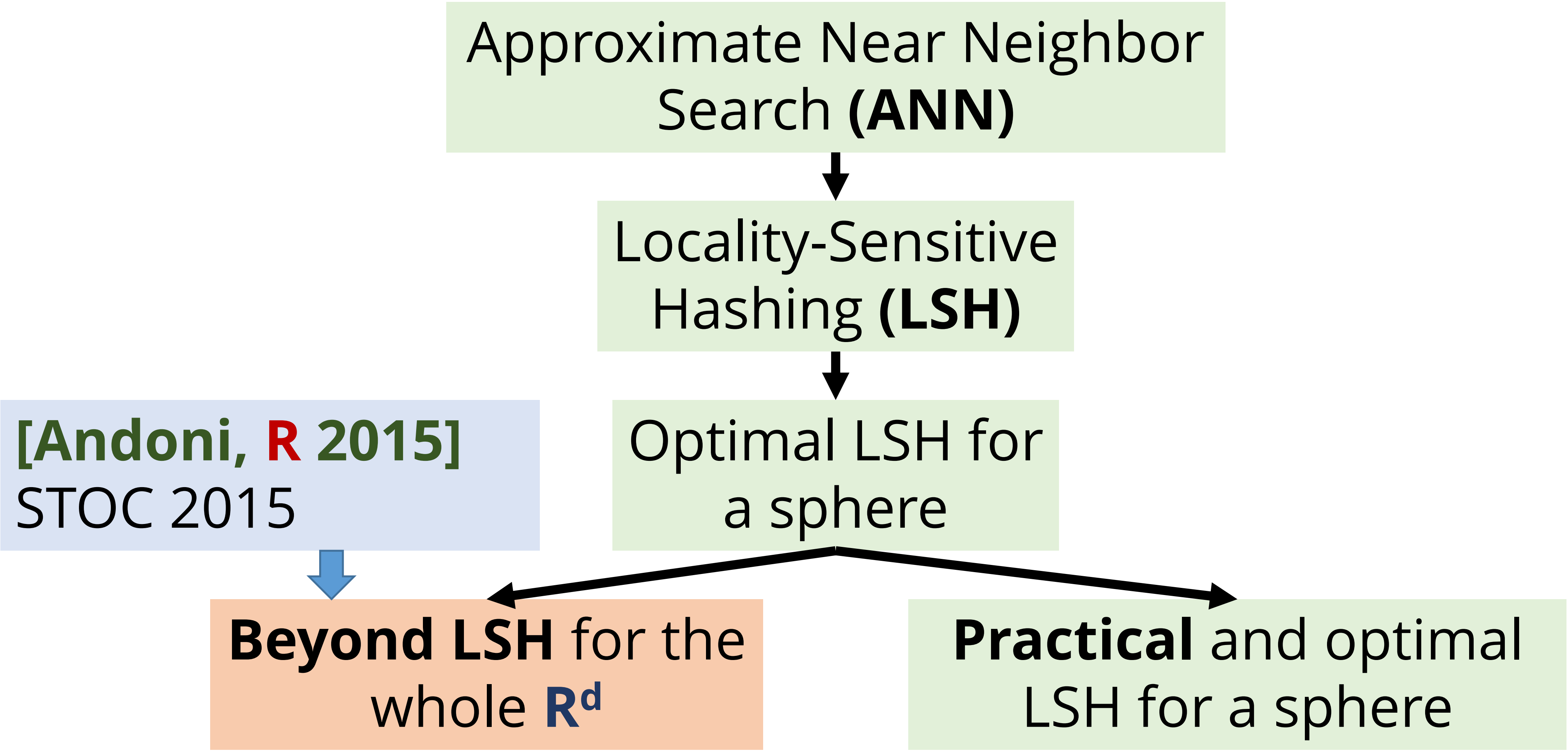
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    - If $\|p - q\| \leq r$, then $\Pr_R[R(p) = R(q)] \geq p_1$
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- **Too strong! Can assume that p is a data point!**
  - Exploit the geometry of **P** to design better partitions
  - Able to obtain **improvement for every P**

The result

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Optimal* data-dependent space partitions for the Euclidean and Manhattan/Hamming distances

* After proper formalization

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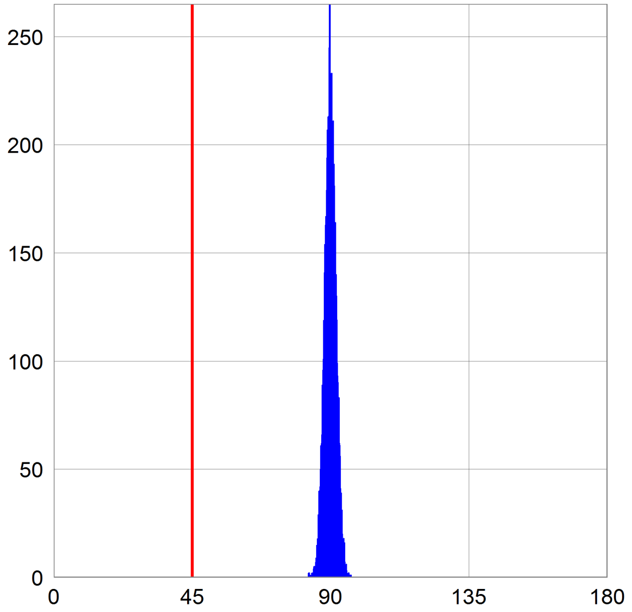

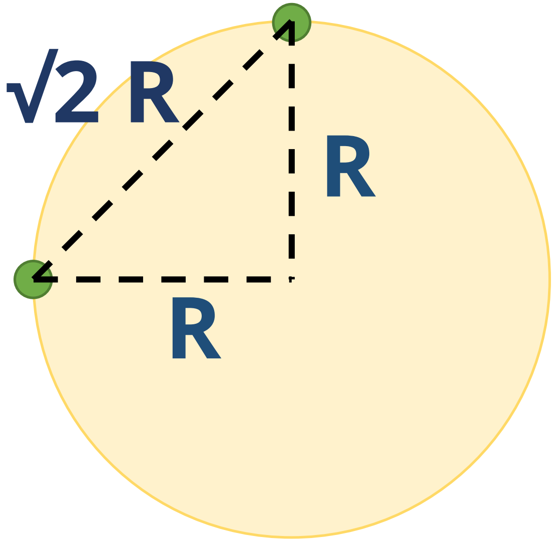

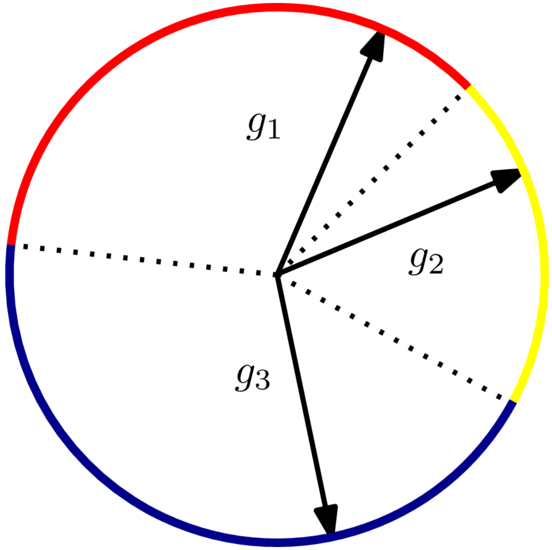

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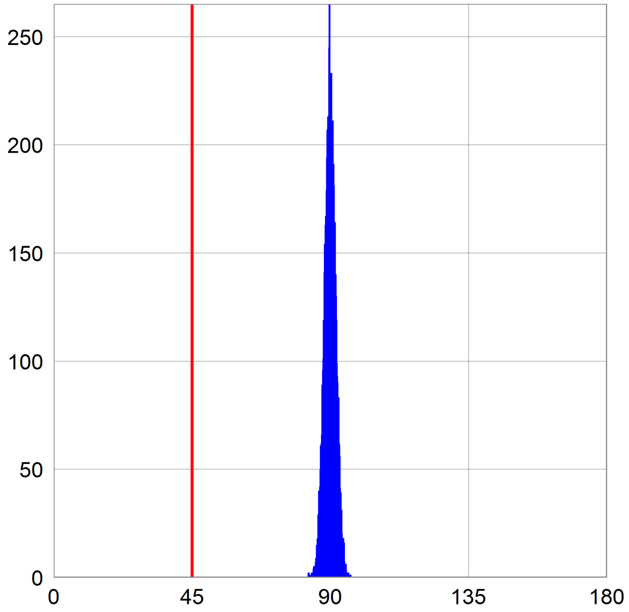

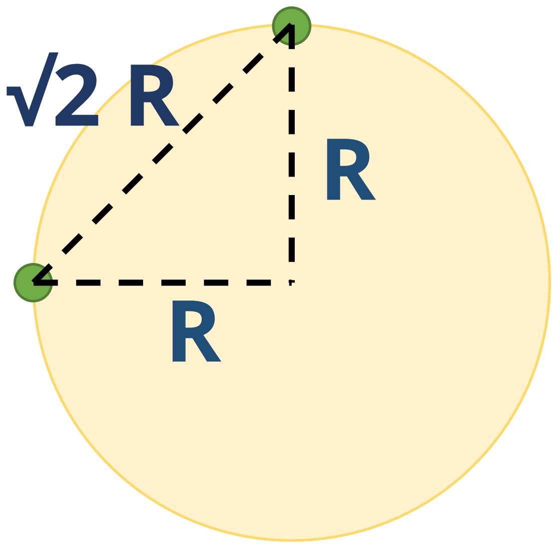

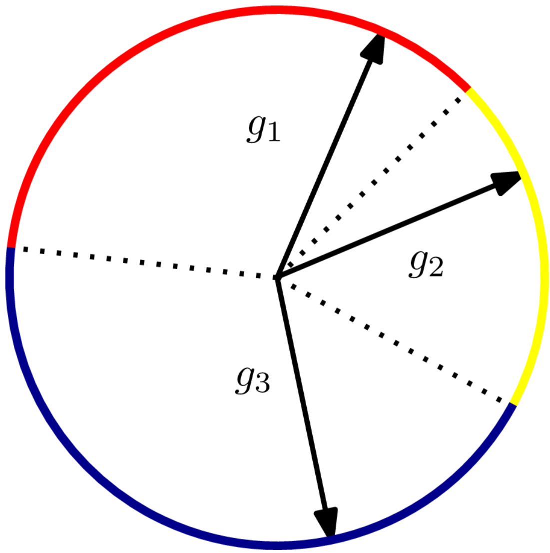

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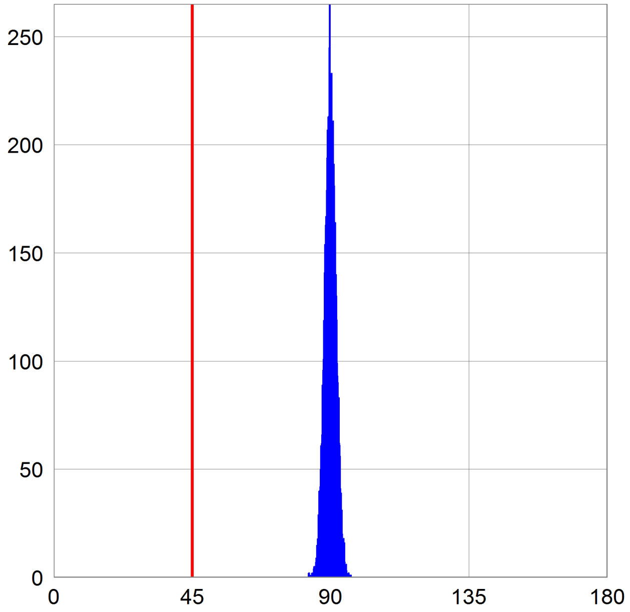
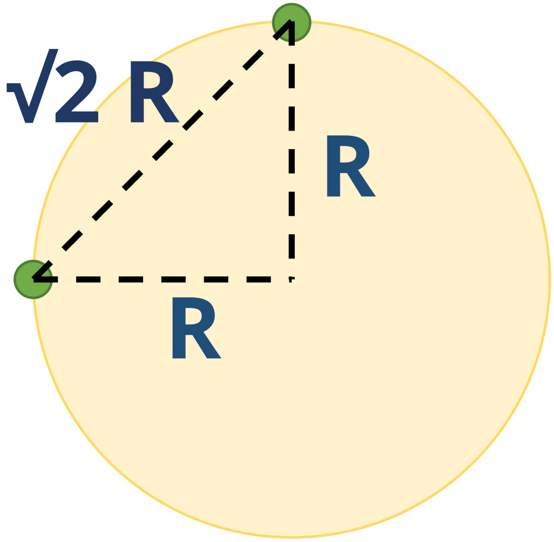
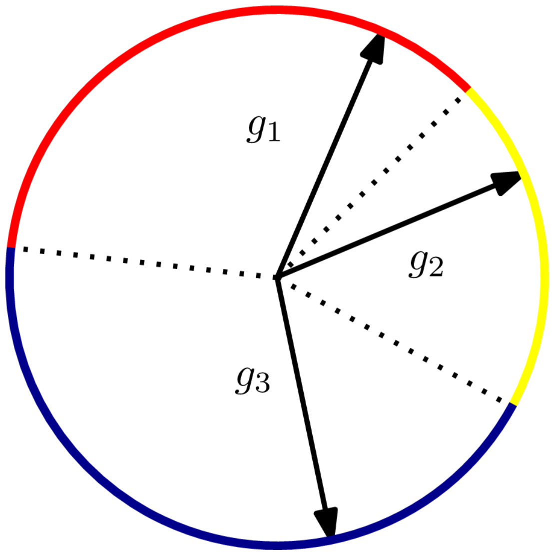
- **What if the dataset does not look random?**
  - Voronoi LSH is suboptimal







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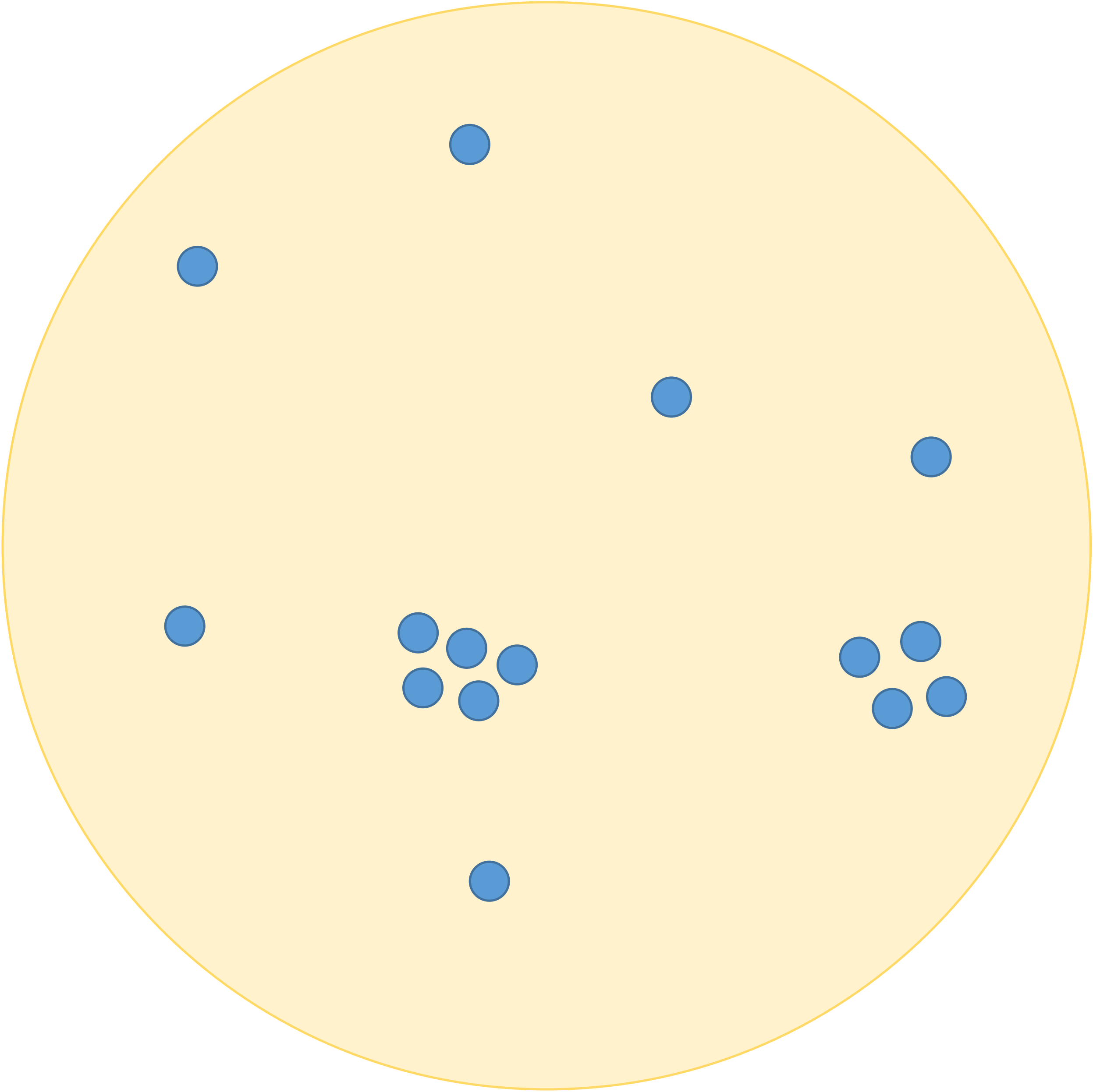

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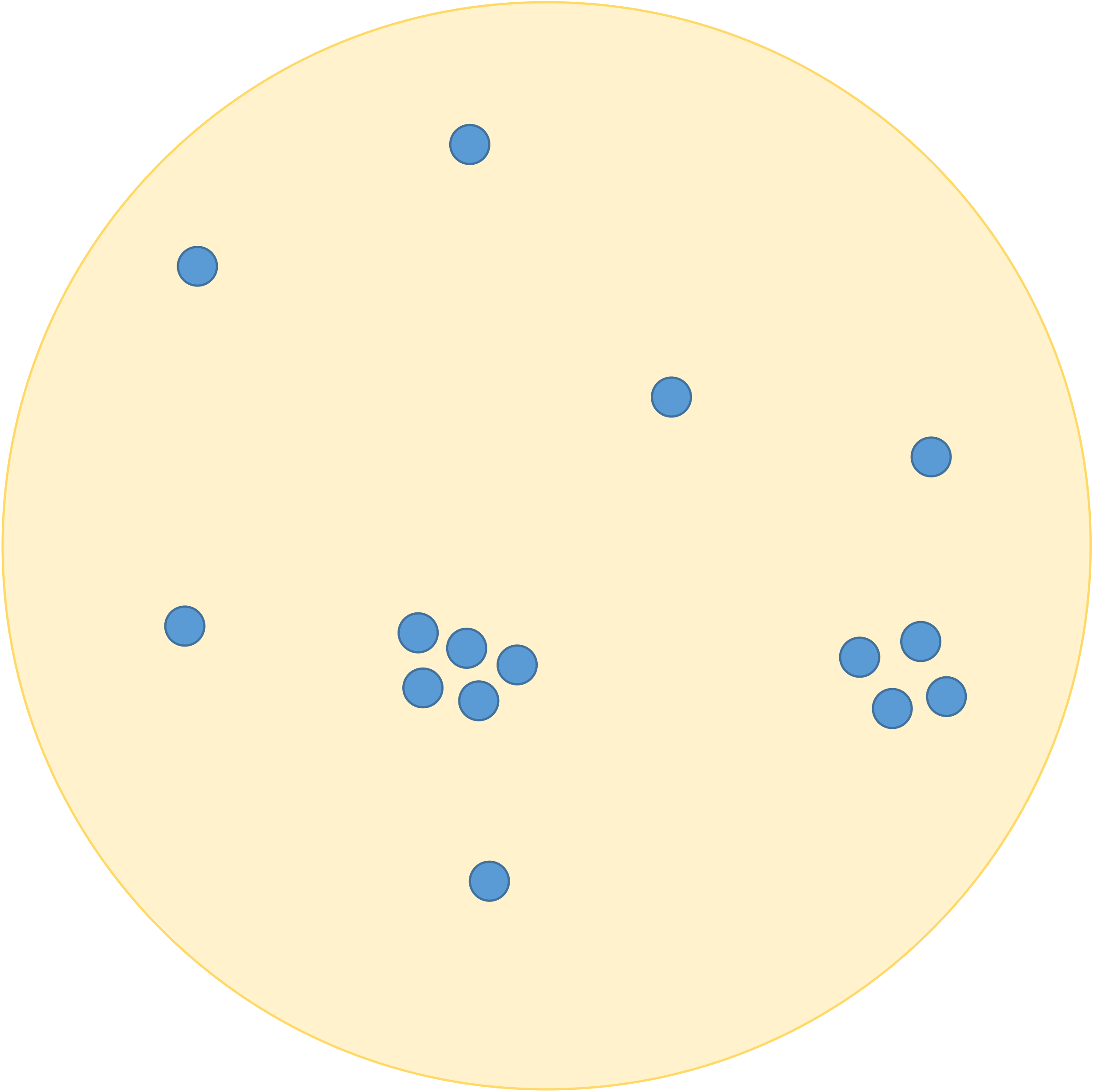

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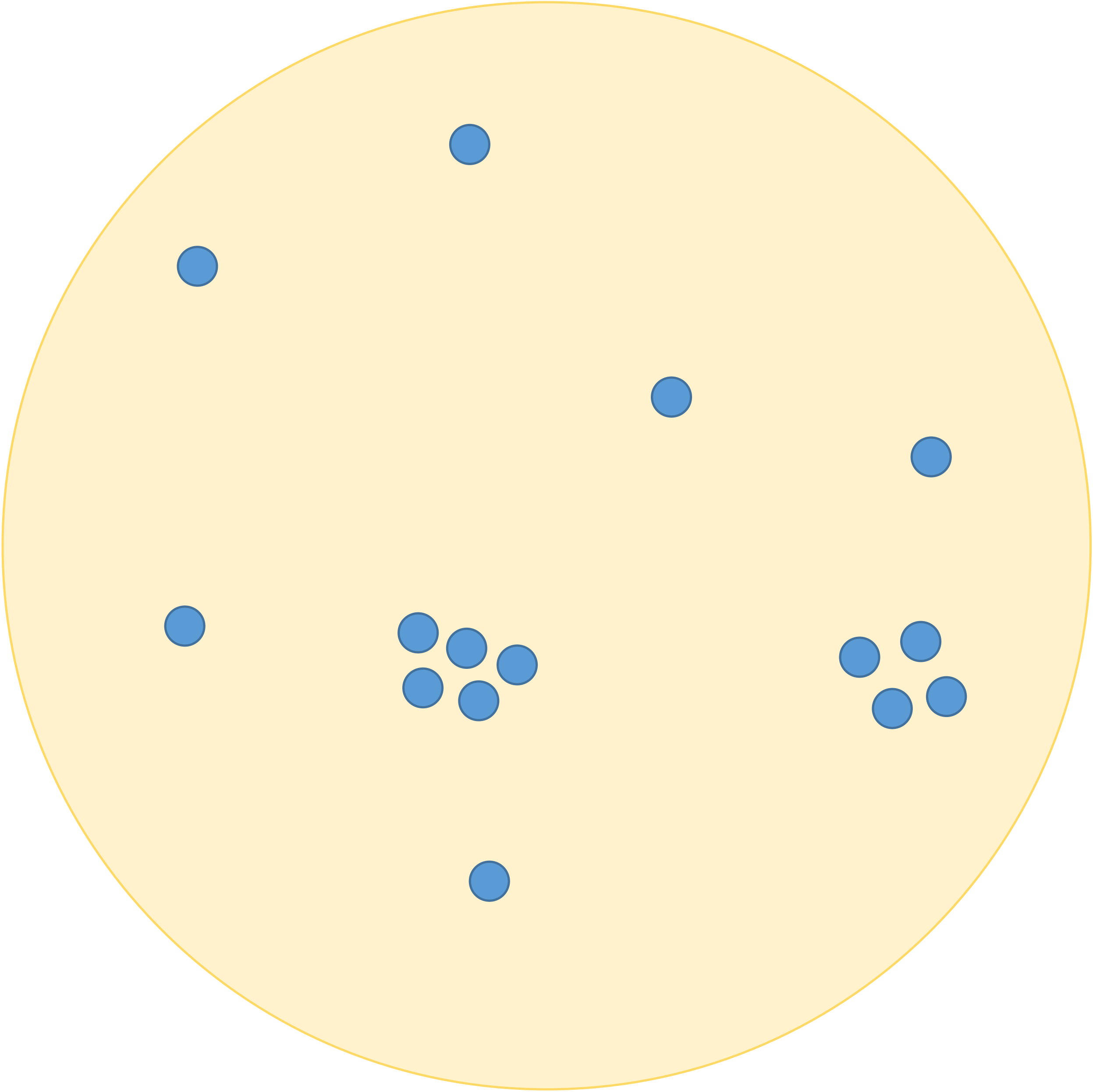

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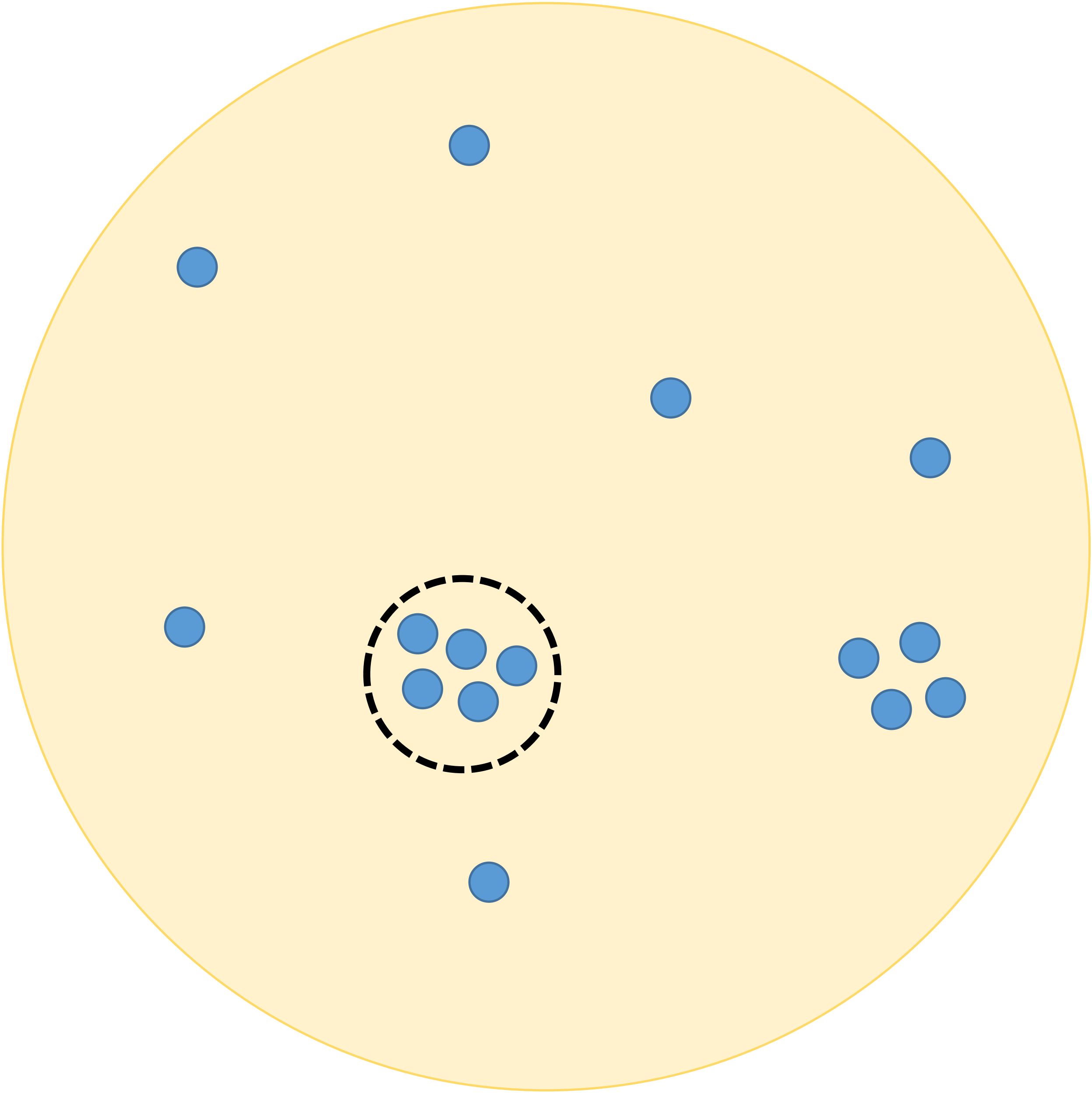

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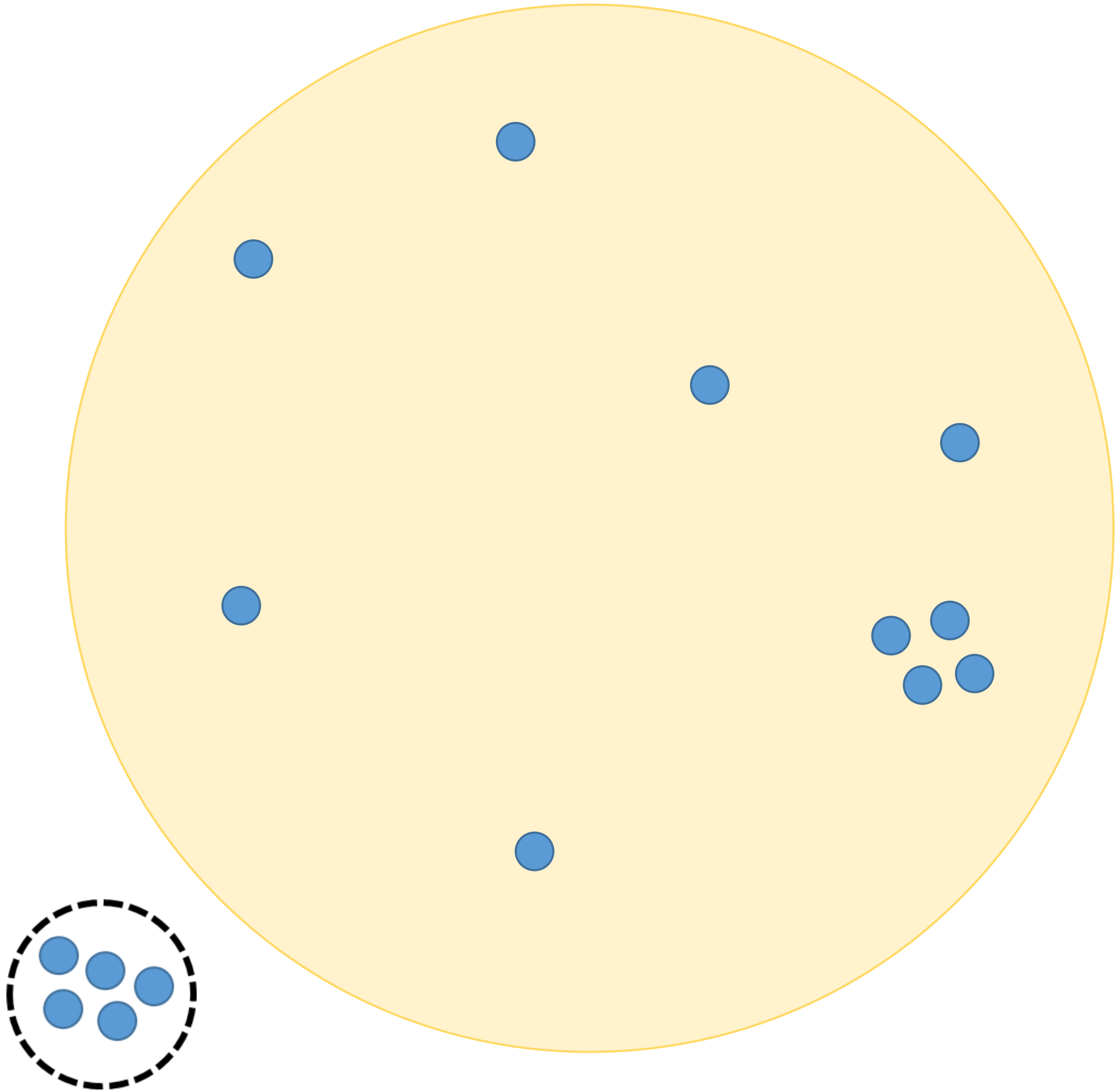

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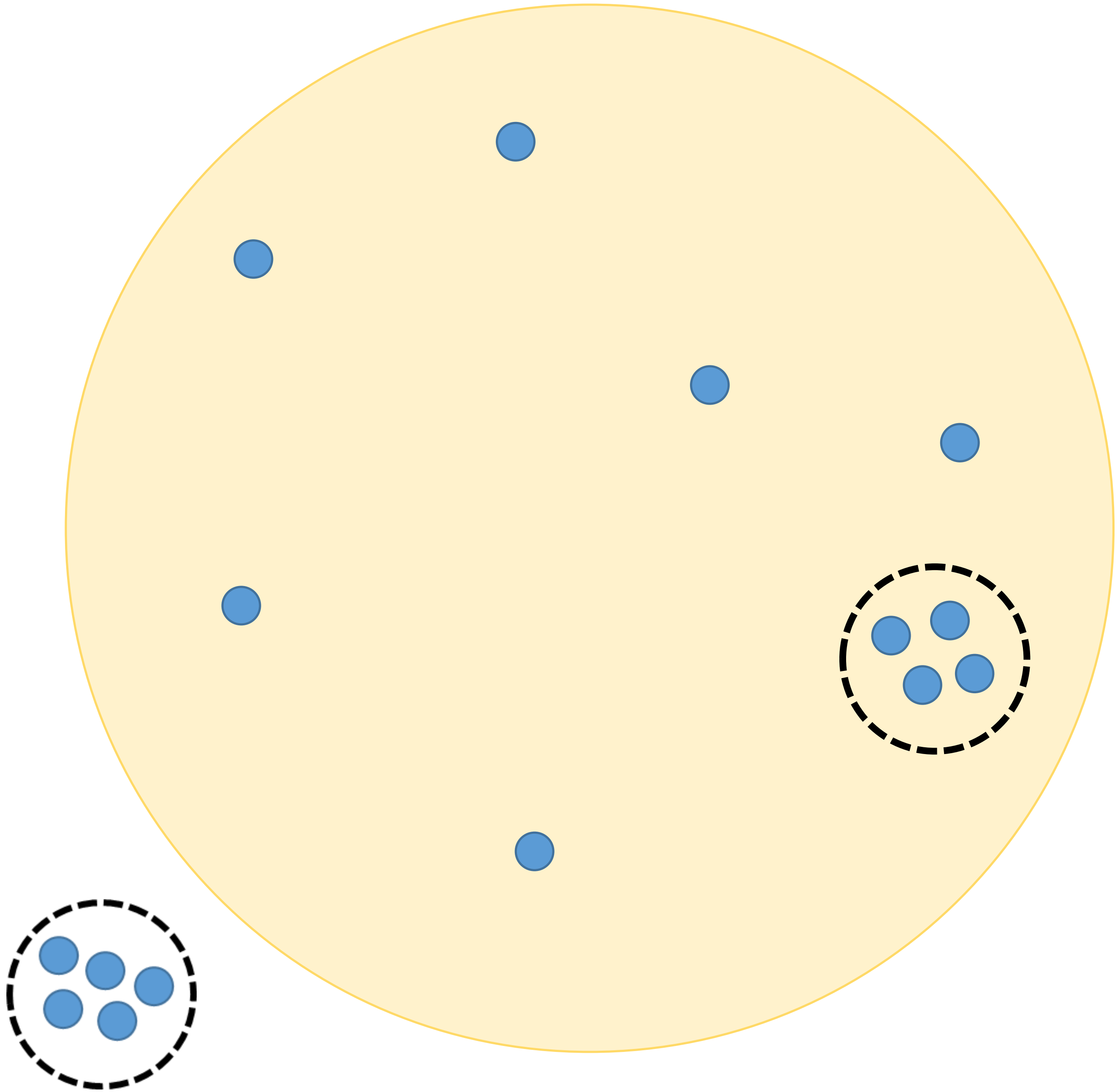

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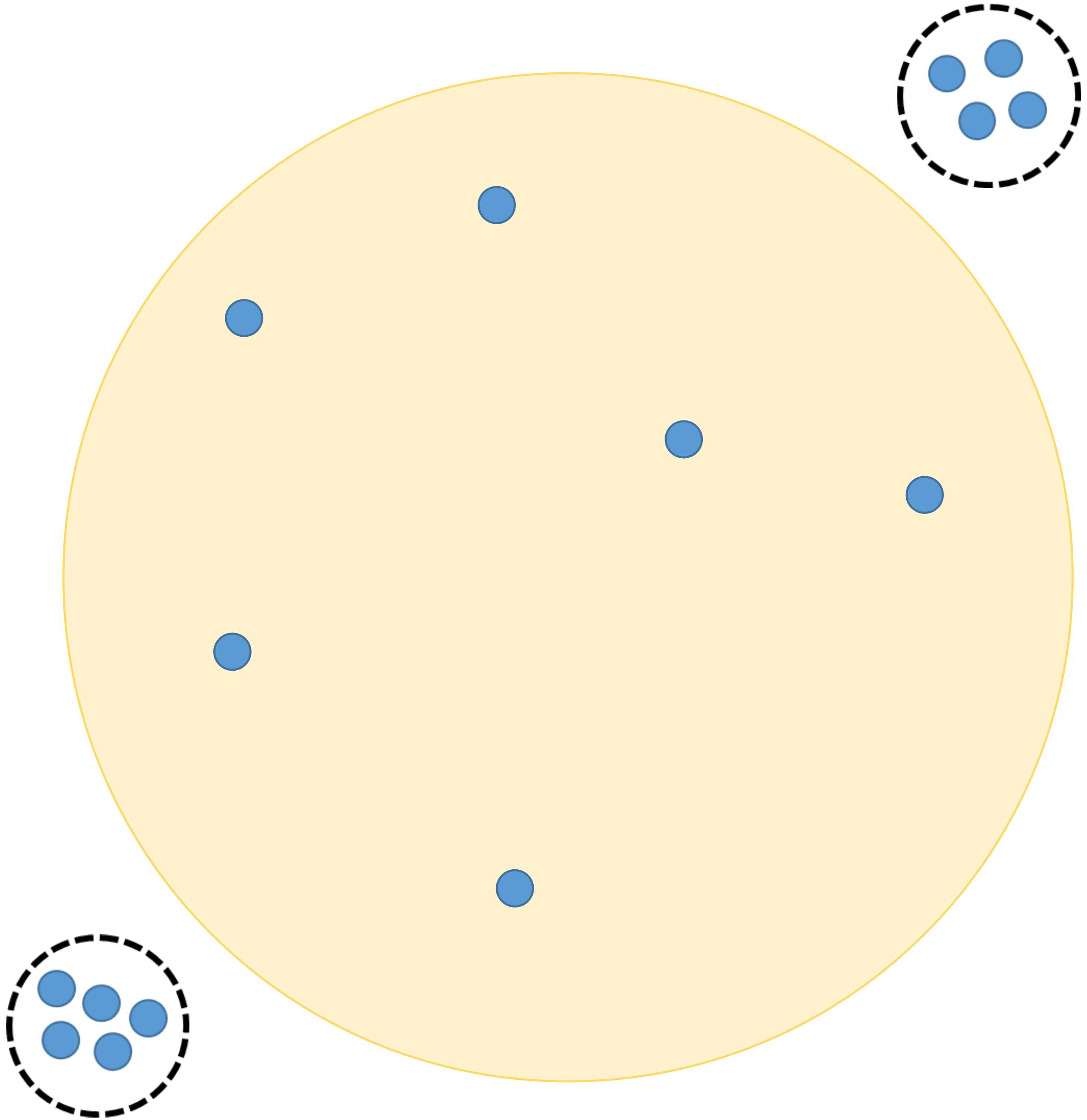

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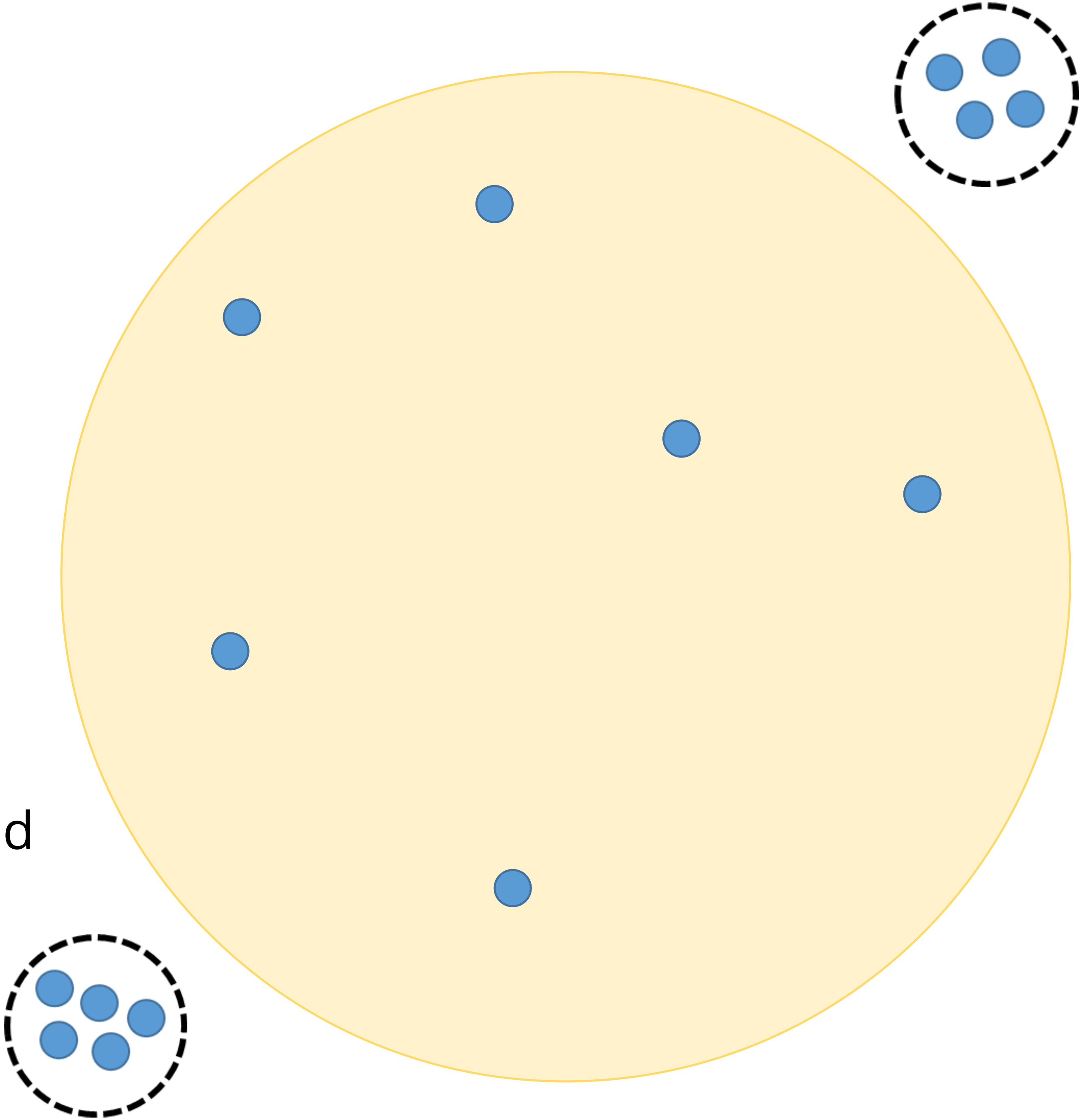

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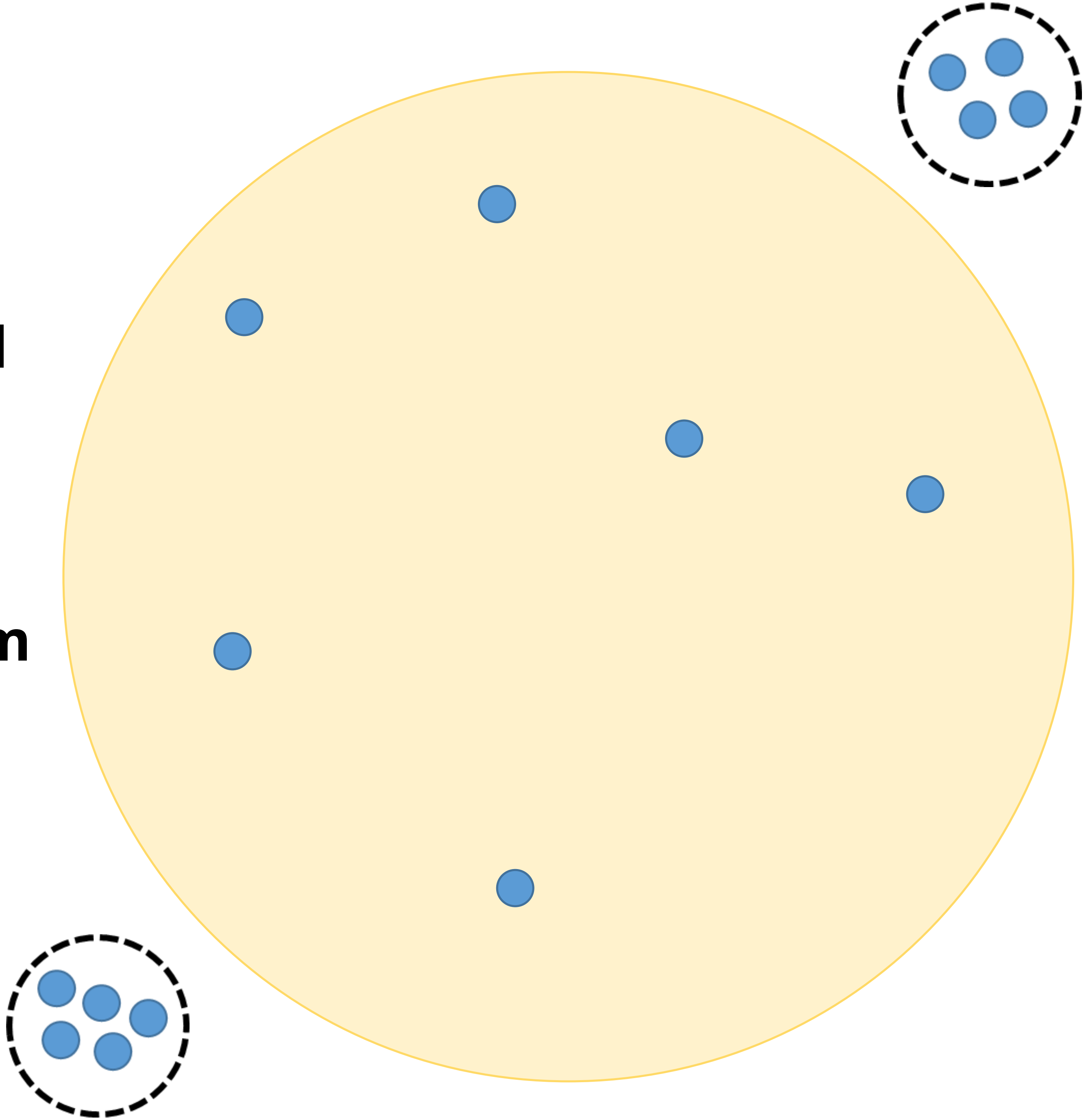

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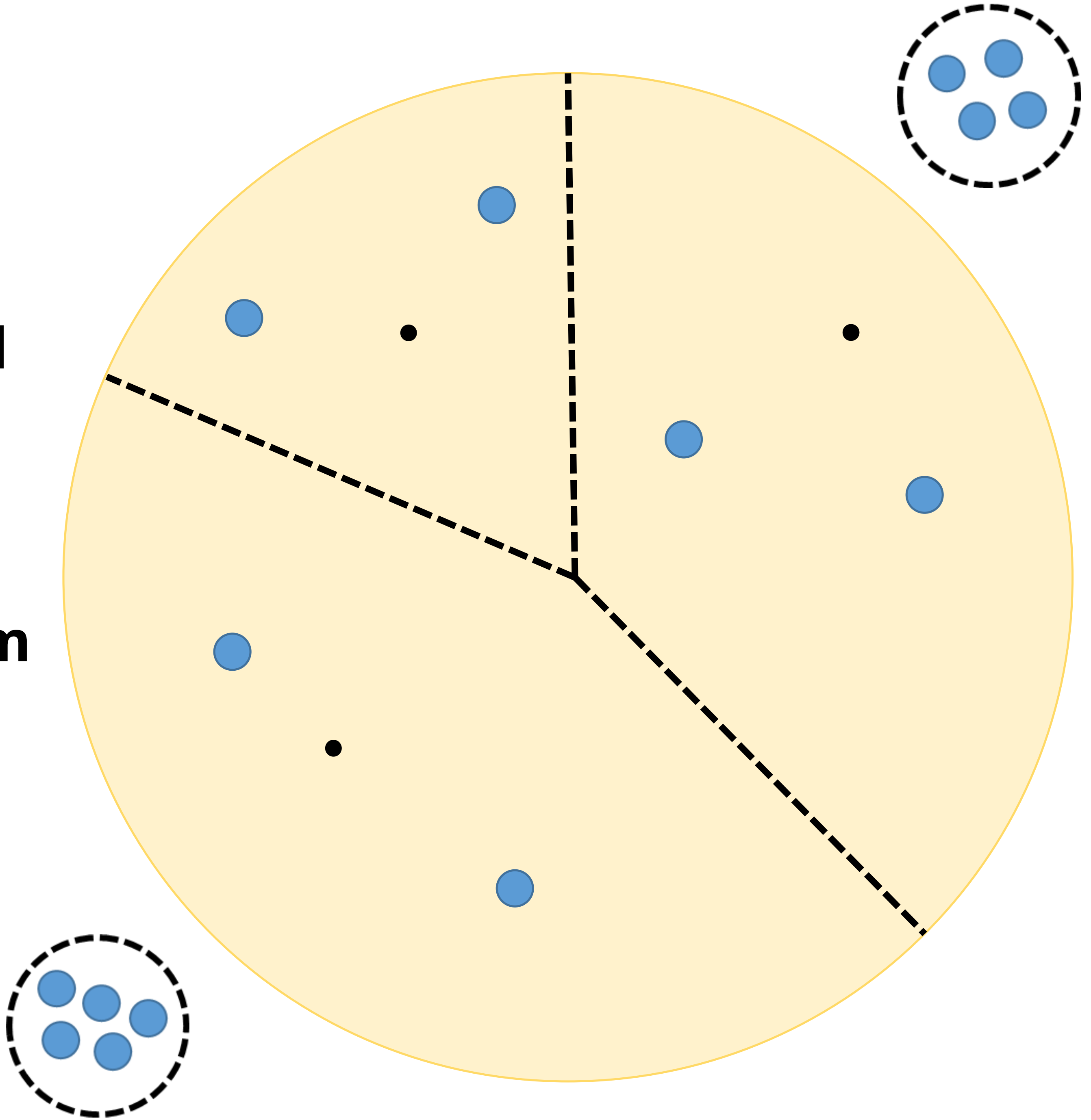

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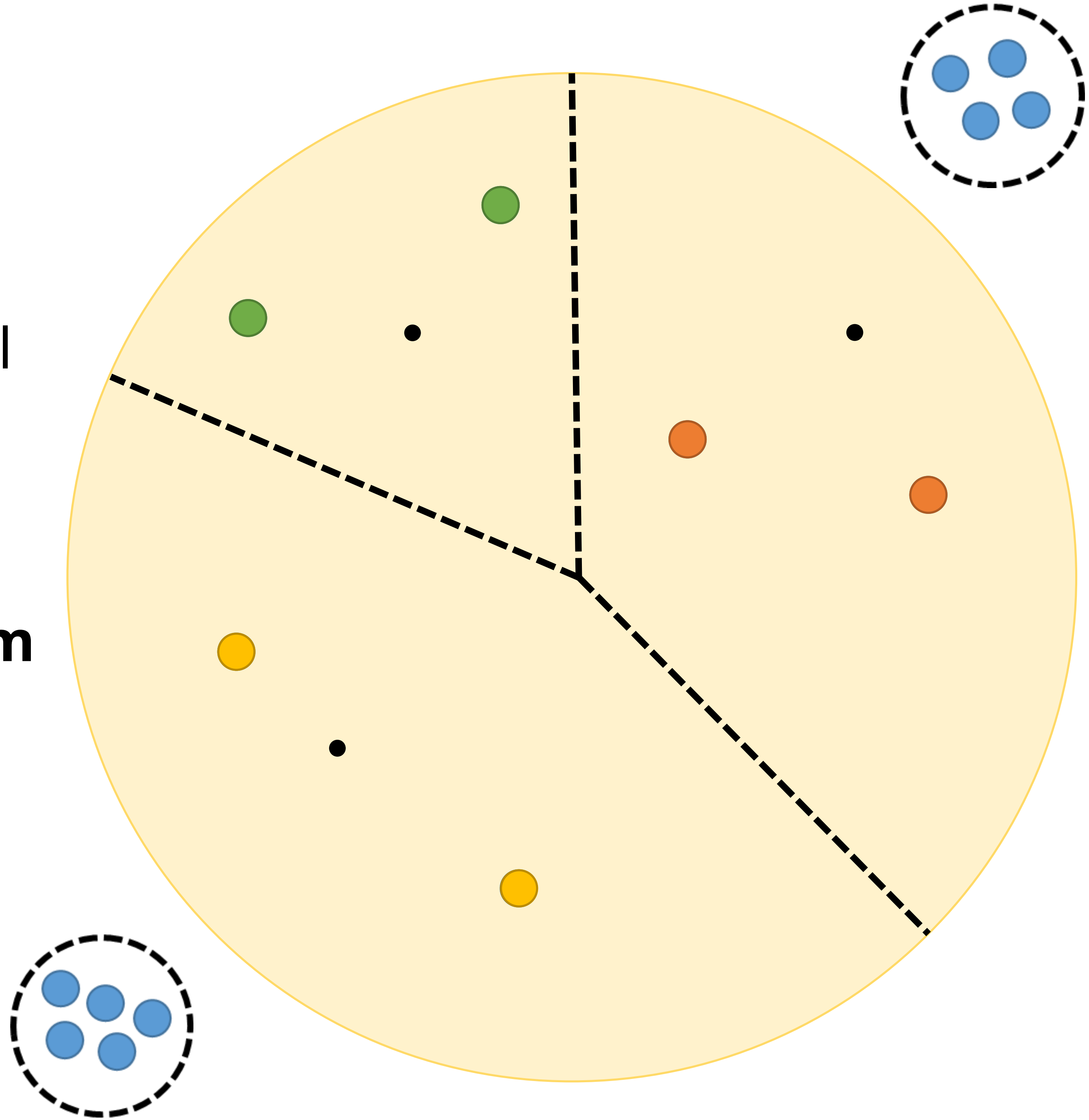

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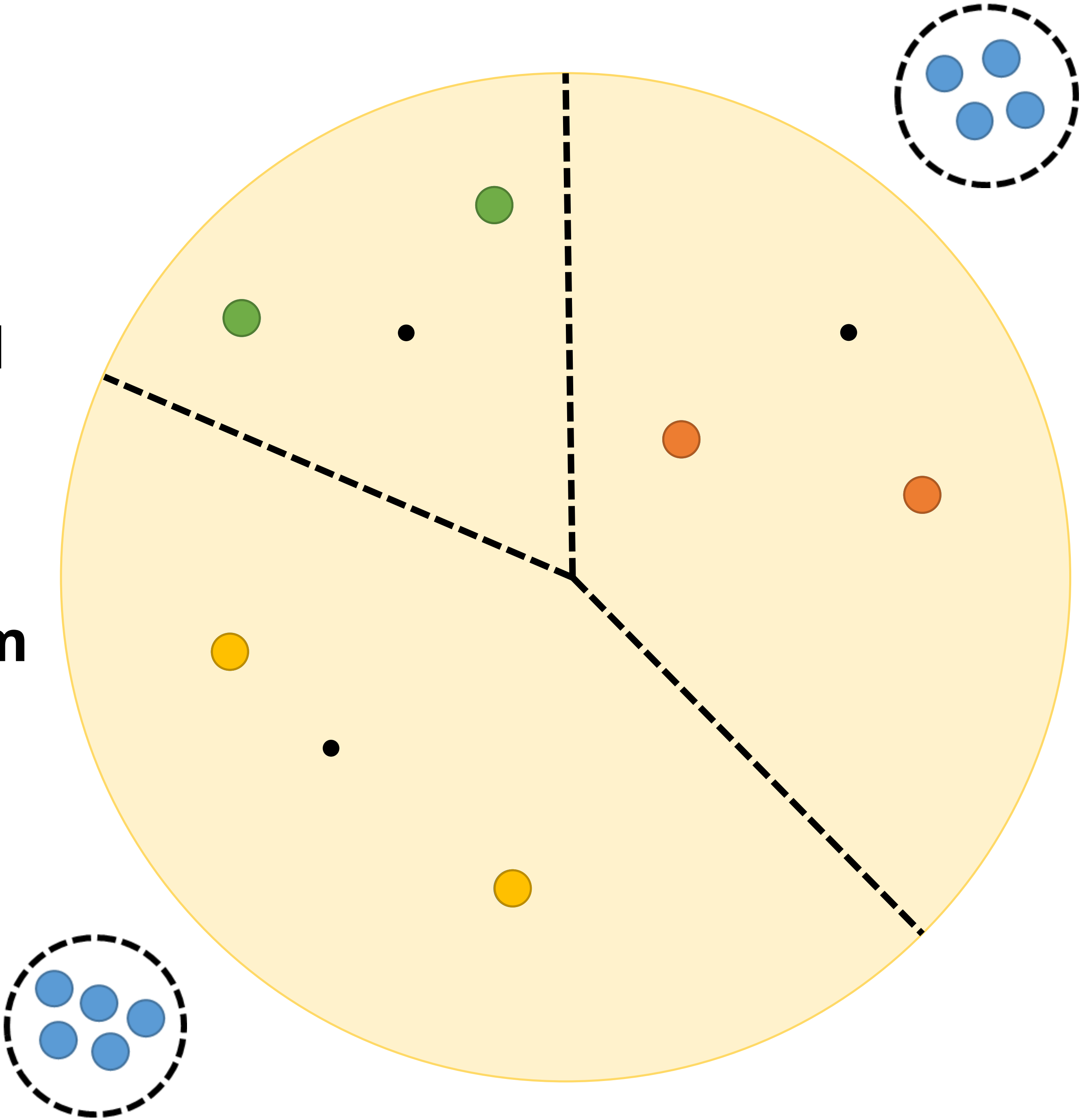
- Query **all** the clusters and **one** part



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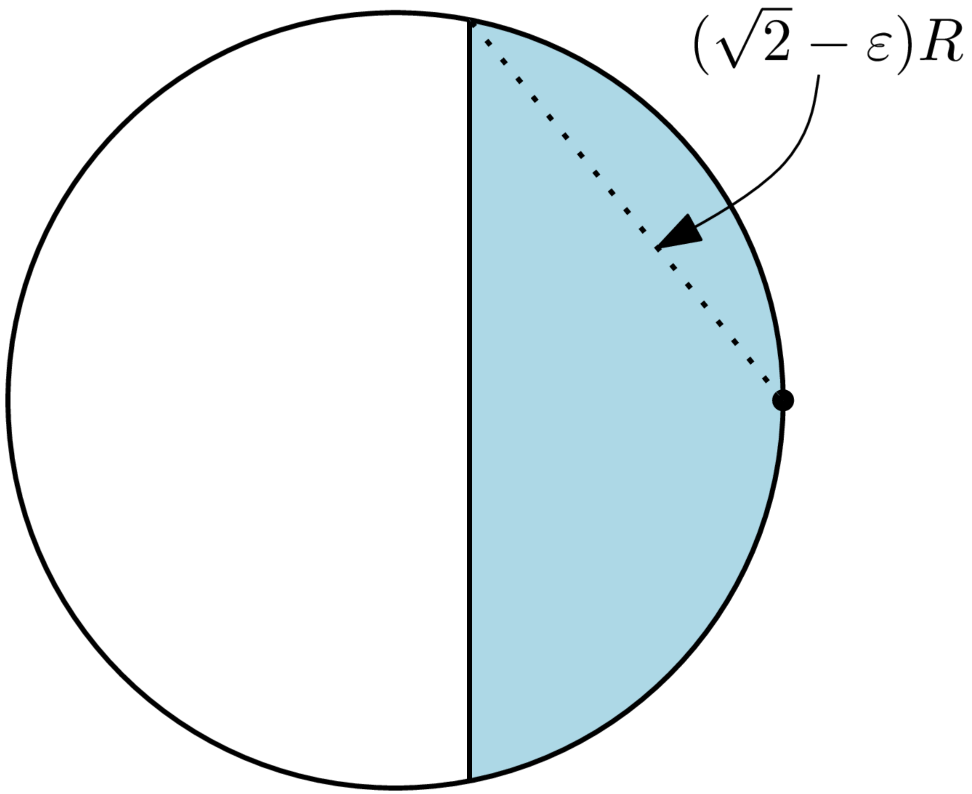

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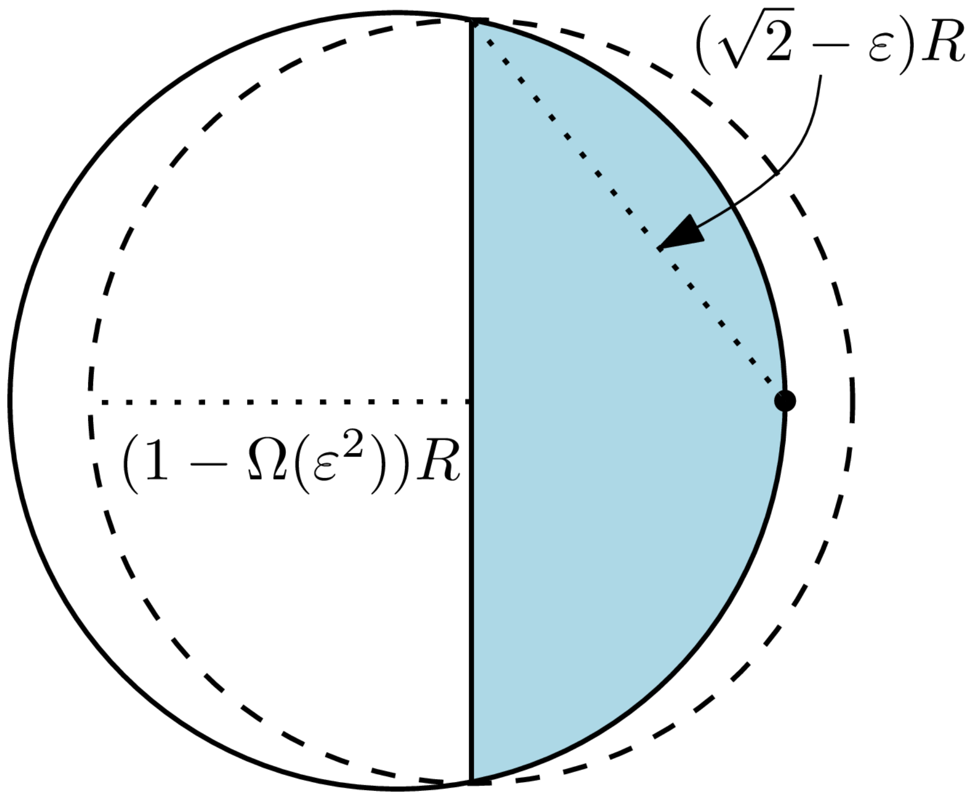

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- Enclose a cluster of radius **$(\sqrt{2} - \varepsilon)R$** in a ball of radius **$(1 - \Omega(\varepsilon^2))\,R$**
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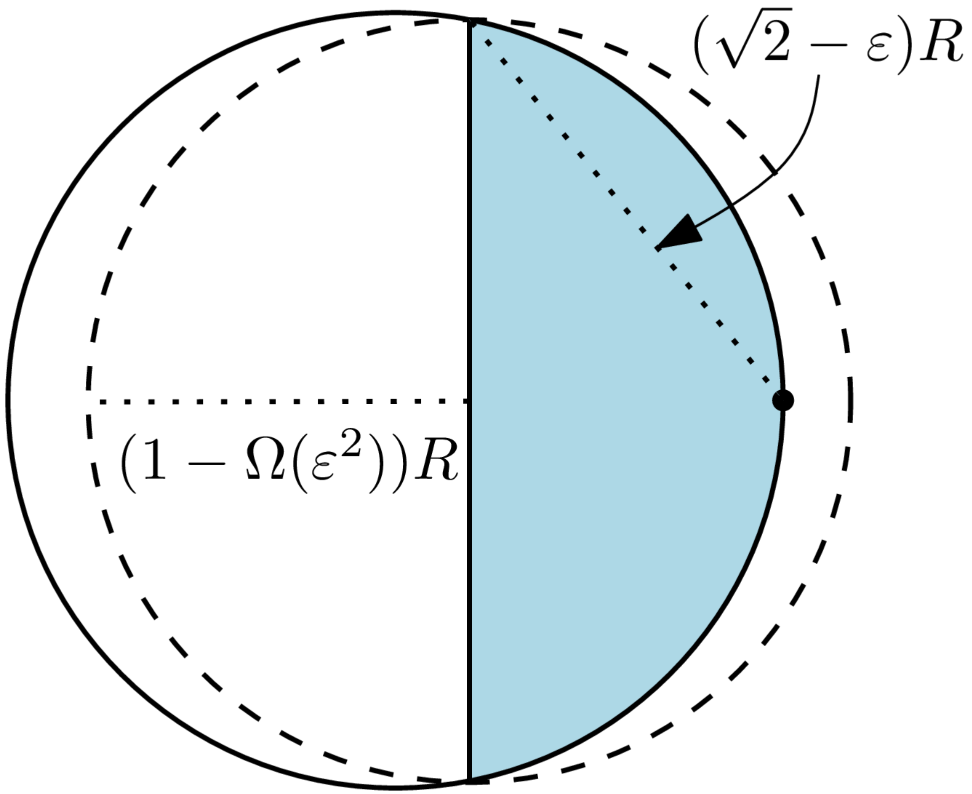

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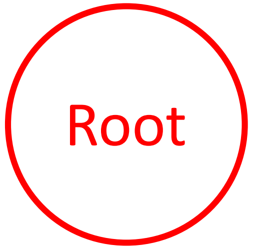

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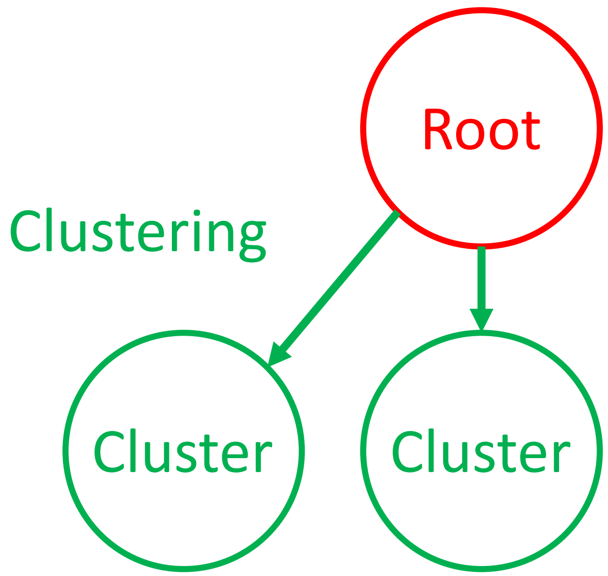

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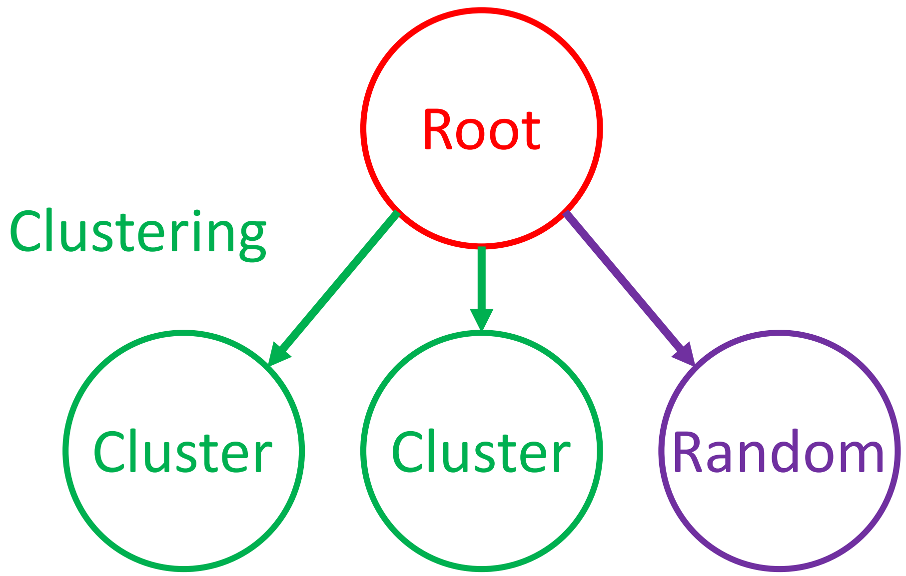

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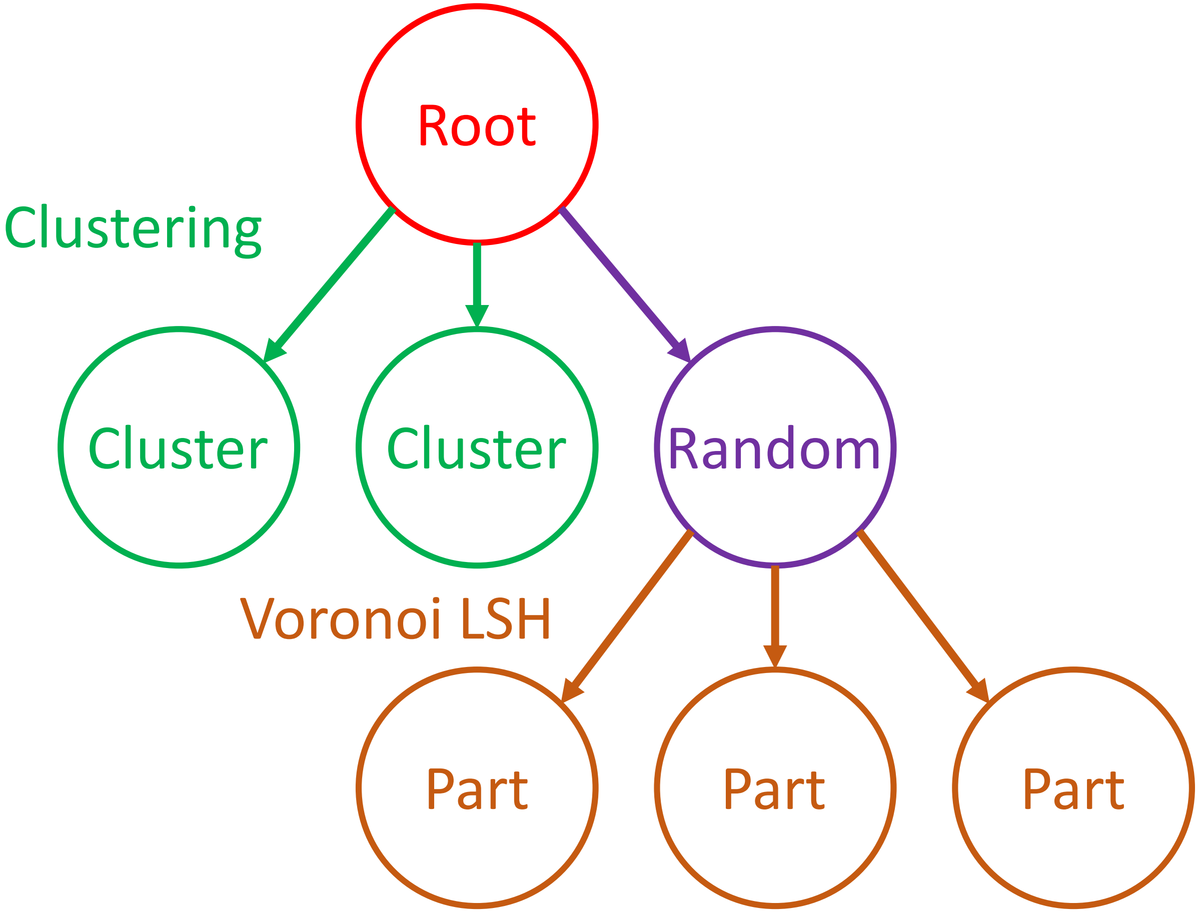

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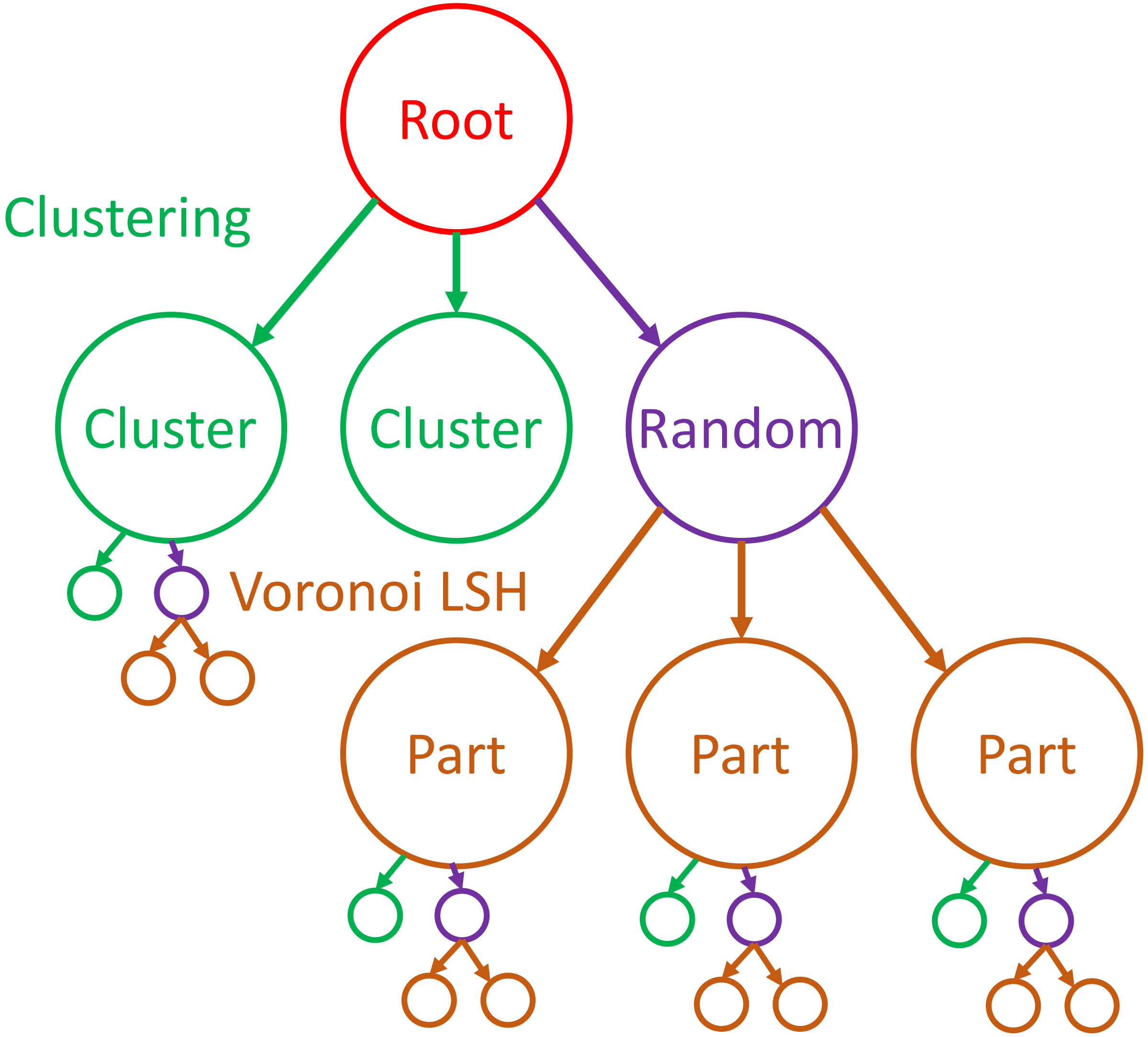

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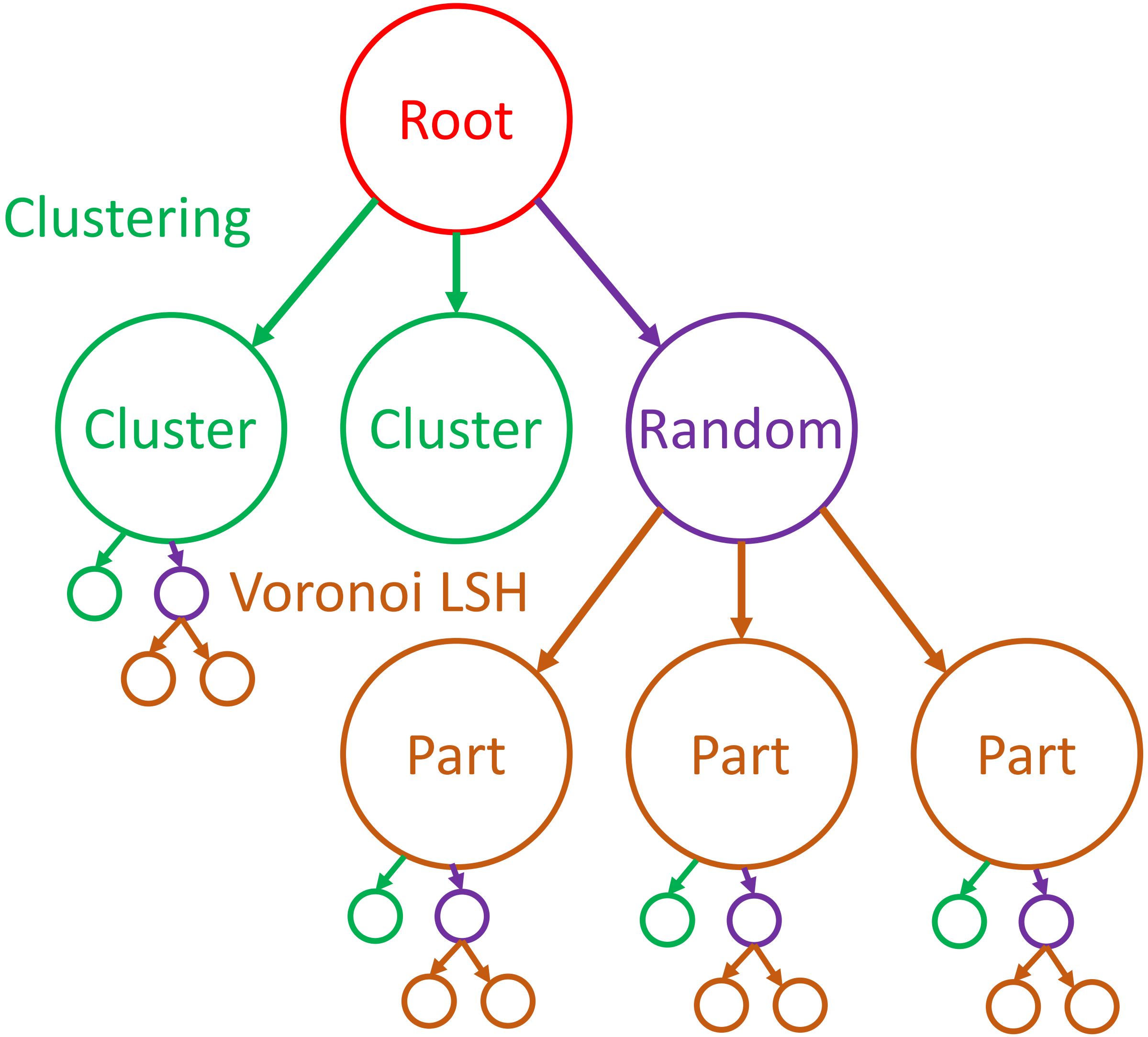

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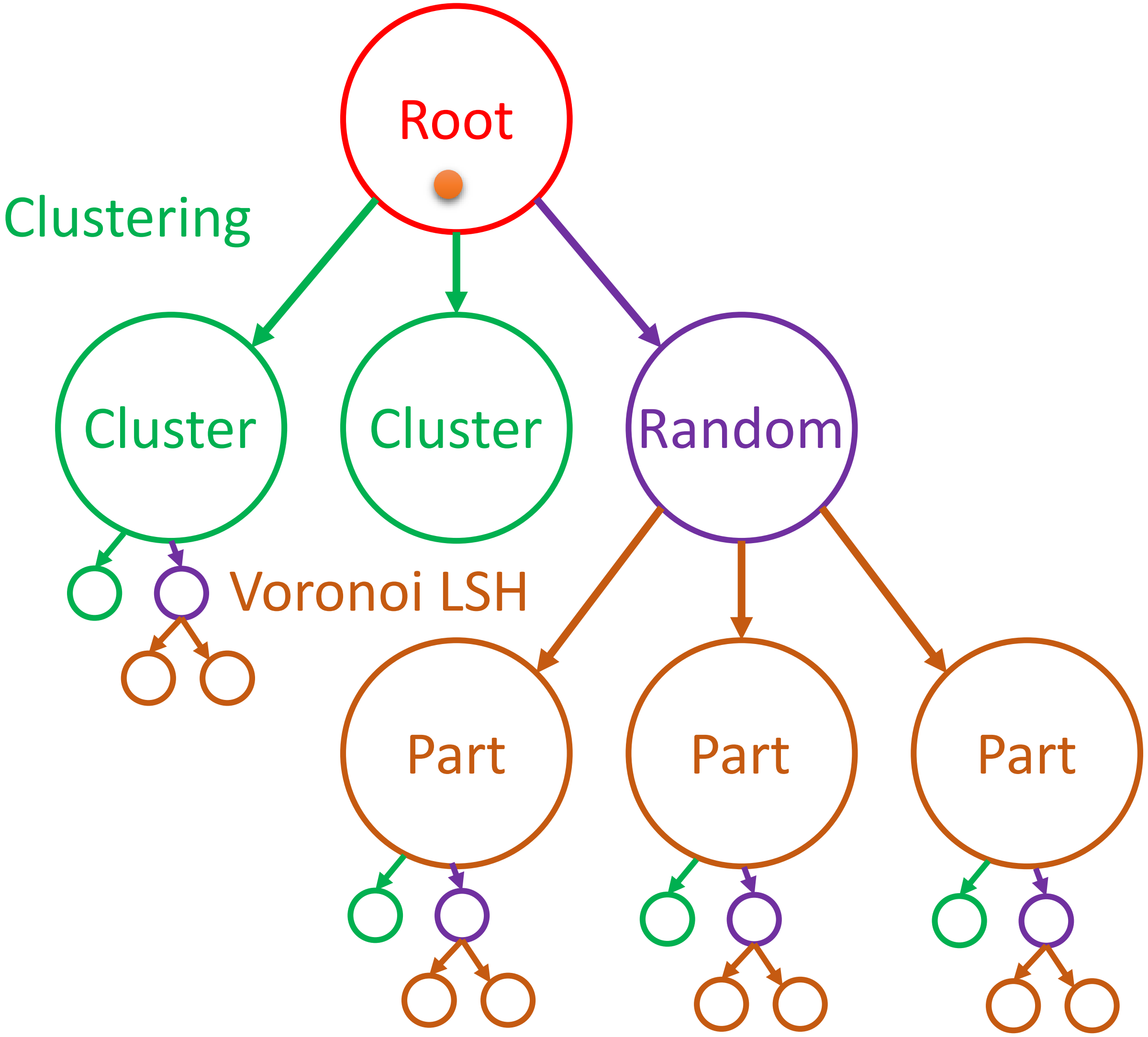

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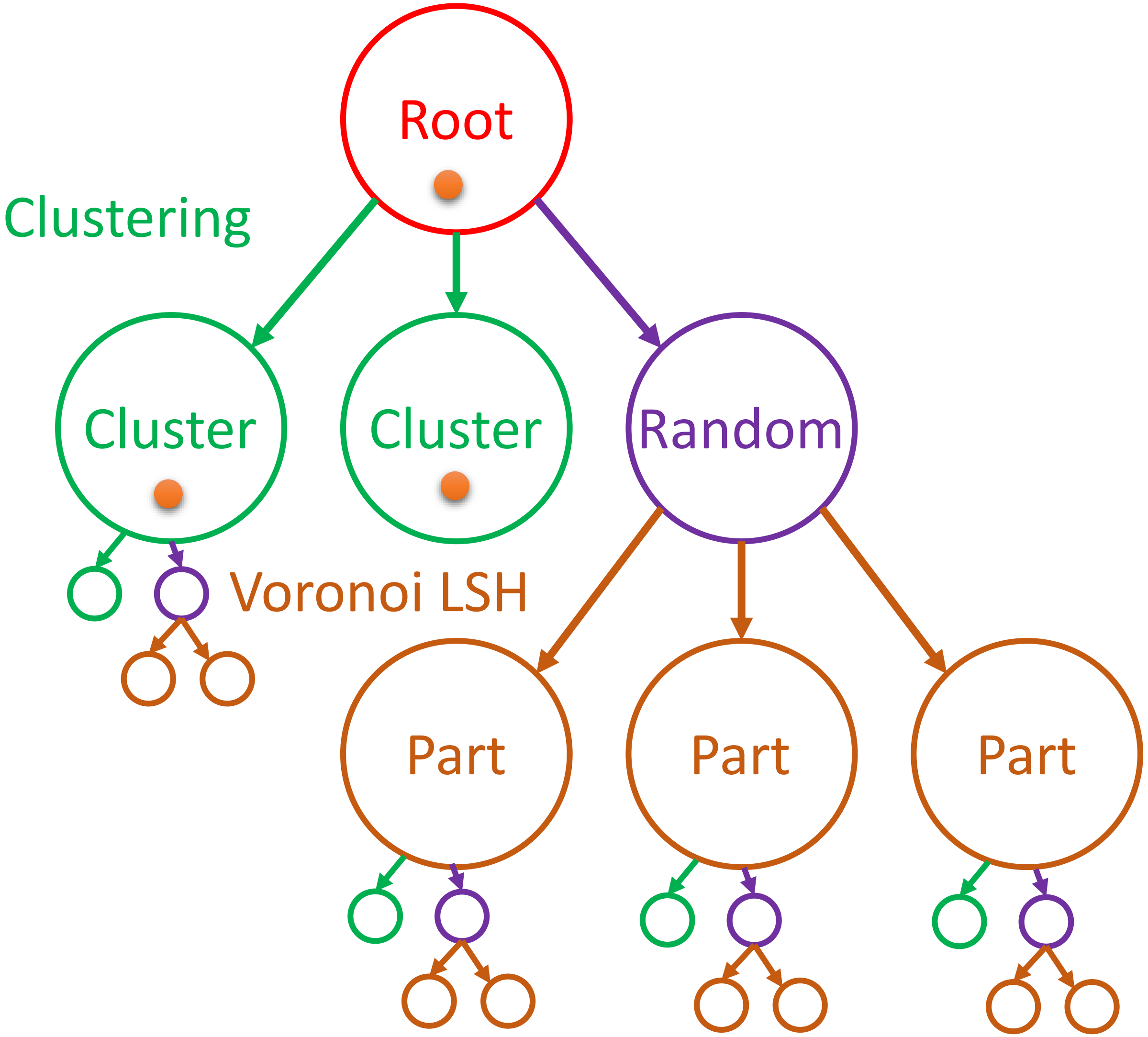

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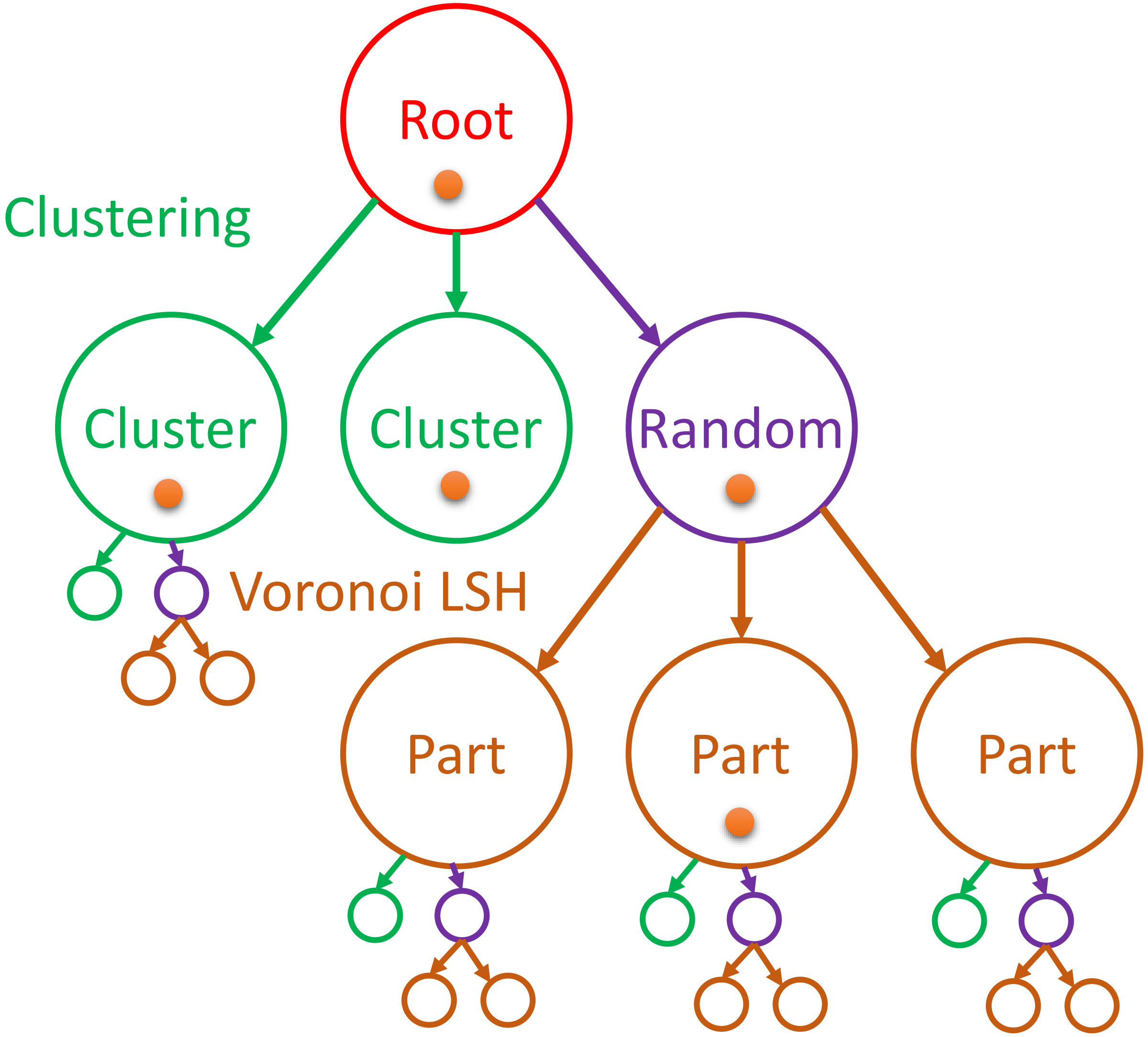

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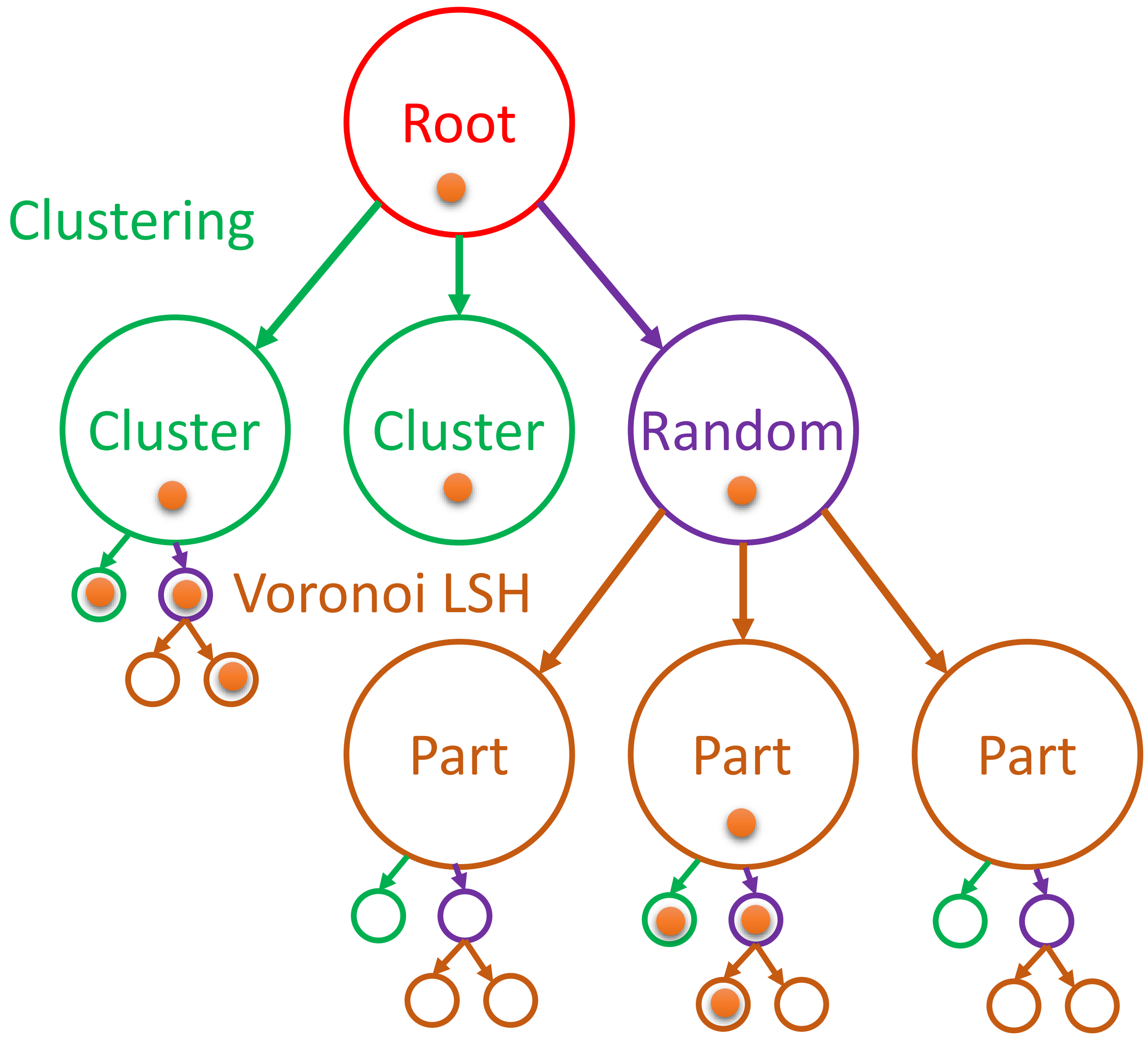

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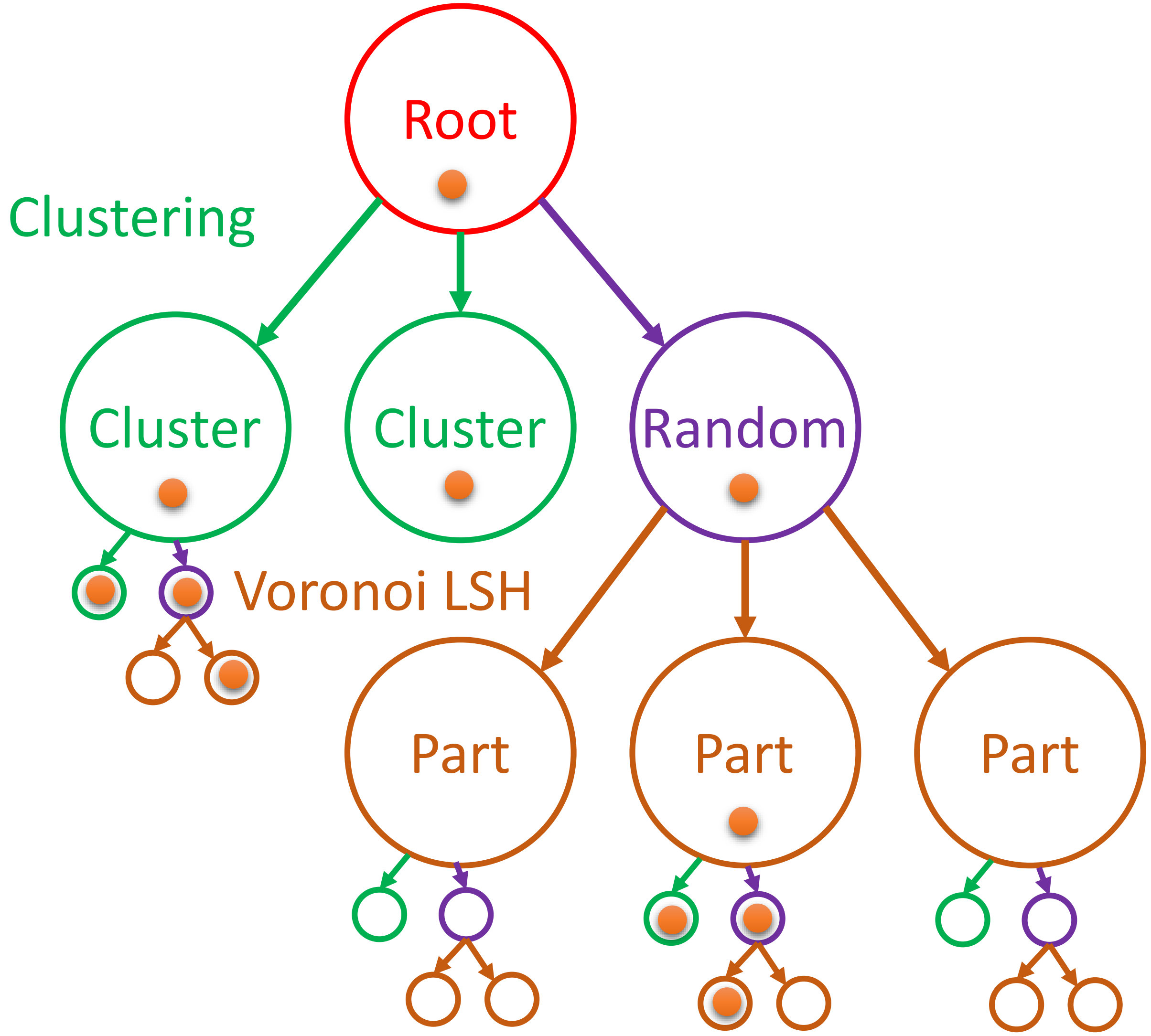
  - Need $\mathbf{n^{\rho}}$ trees to succeed w.h.p.



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[Andoni, Indyk, Laarhoven, R, Schmidt 2015] NIPS 2015

Practical and optimal LSH for a sphere

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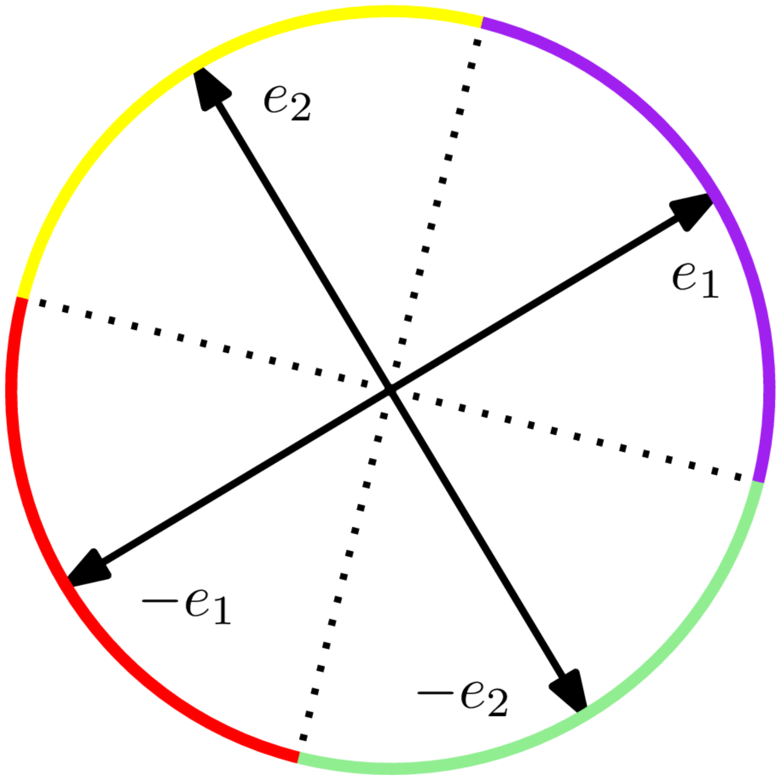

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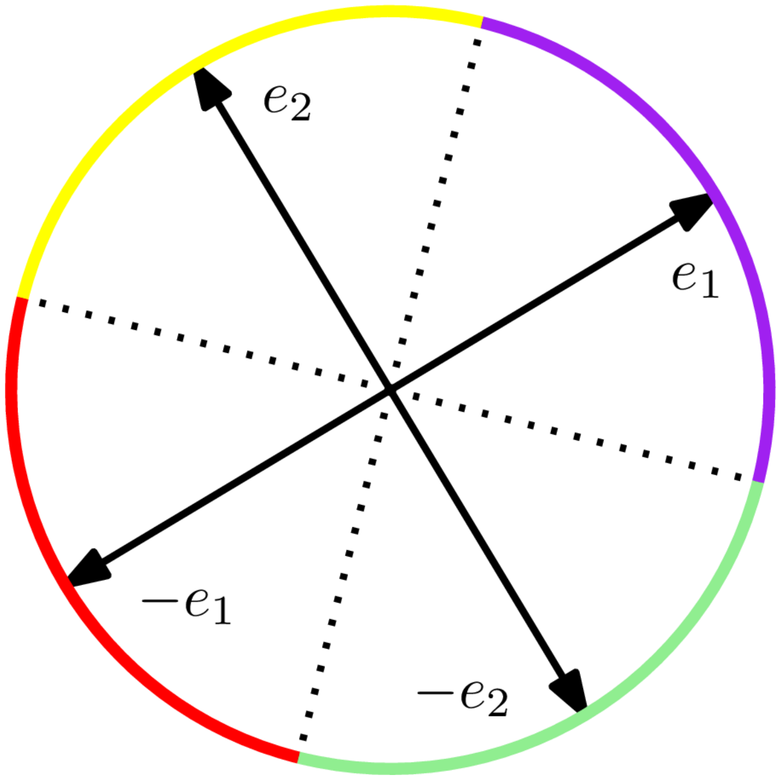

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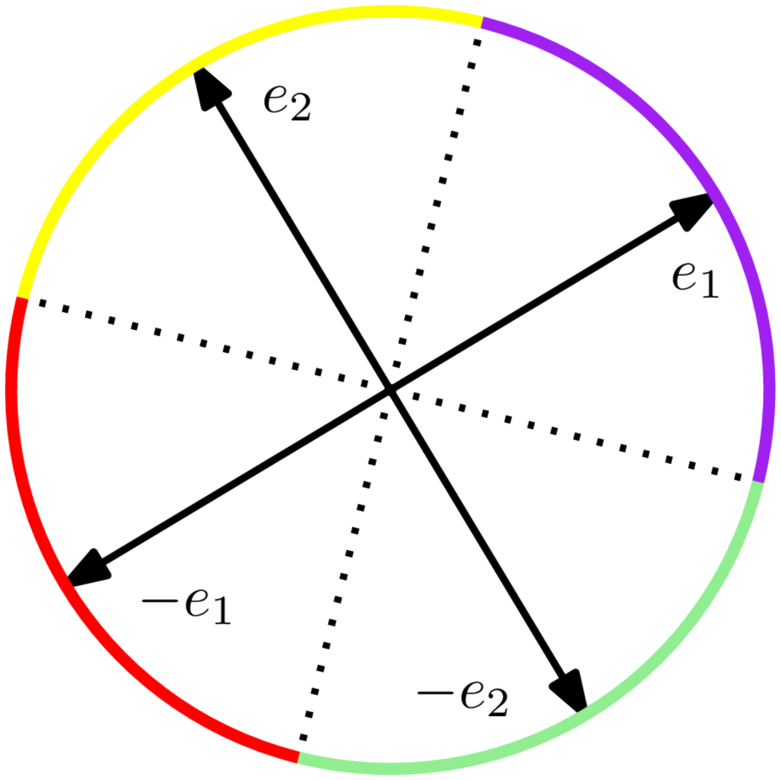

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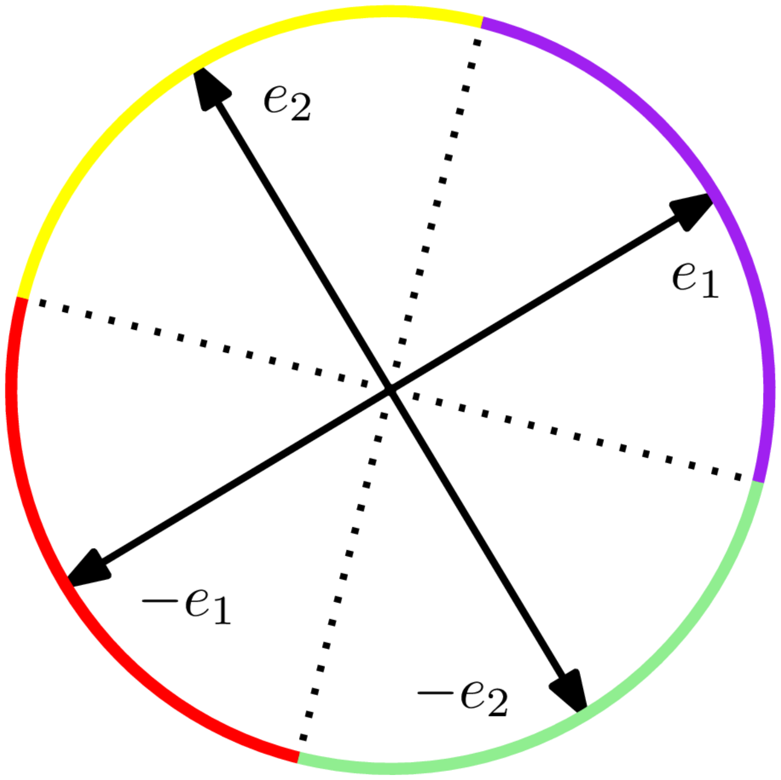

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↓ **Hadamard**

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