Convex Optimization

(EE227A: UC Berkeley)

Lecture 5
(Optimization problems)

05 Feb, 2013

Suvrit Sra

Organizational

- Homeworks due in class: 2/14/2013
- No late homeworks will be accepted
- Team up for projects into groups of 3–4 (max)
- Talk to me if special concerns
- We're using Piazza for Q/A — sign up!
- Bspace has the rest (course material, links, etc.)

Challenge

Consider the following functions on strictly positive variables:

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\begin{aligned}
h_1(x) &:= \frac{1}{x} \\
h_2(x,y) &:= \frac{1}{x} + \frac{1}{y} - \frac{1}{x+y} \\
h_3(x,y,z) &:= \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - \frac{1}{x+y} - \frac{1}{y+z} - \frac{1}{x+z} + \frac{1}{x+y+z}
\end{aligned}
$$

- ♡ Prove that h_1, h_2, h_3, and in general h_n are convex!
- ♡ Prove that in fact each $1/h_n$ is concave
- ♡ Generalize to where denom. replaced by $g(x)$, $g(x+y)$, $g(x+y+z)$, etc.

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$\nabla^2 h_n(x) \succeq 0$ is not recommended 🙂

Optimization problems

Let $f_i : \mathbb{R}^n \to \mathbb{R}$ $(0 \le i \le m)$. Generic **nonlinear program**

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \le 0, \quad 1 \le i \le m, \\ & x \in \{\mathrm{dom}\, f_0 \cap \mathrm{dom}\, f_1 \cdots \cap \mathrm{dom}\, f_m\}\,. \end{aligned}$$

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- If f_i are **differentiable** — smooth optimization
- If any of the f_i is **non-differentiable** — nonsmooth optimization
- If all f_i are **convex** — convex optimization
- If $m = 0$, i.e., only f_0 is there — **unconstrained** minimization

Convex optimization problems

Standard form

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- Direction of inequality $f_i(x) \leq 0$ **crucial**
- The only equality constraints we allow are affine
- This ensures, set of feasible solutions is also **convex**

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- Sometimes **minimum doesn't exist** (as $x \to \pm\infty$)
- Say $f_0(x) = 0$, problem is called **convex feasibility**

Optimality

Def. A point $x^* \in \mathcal{X}$ is **locally optimal** if $f(x^*) \le f(x)$ for all x in a **neighborhood** of x^*. **Global** if $f(x^*) \le f(x)$ for **all** $x \in \mathcal{X}$.

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- So rhs is also nonnegative, proving $f(y) \geq f(x^*)$ as desired.

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Exercise: Verify that $\mathcal{X}^*$ is always convex.

First-order optimality conditions

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Exercise: Prove that if f is convex, then $\nabla f(x^*) = 0$ is actually **sufficient** for global optimality! For general f this is **not** true. (This property that makes convex optimization special!)

Optimality conditions – constrained

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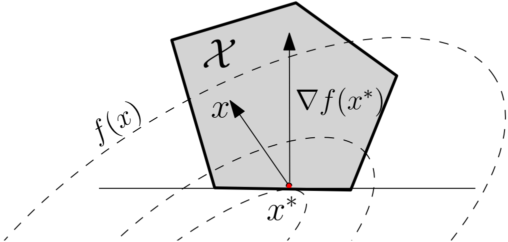
♠ If $\mathcal{X} = \mathbb{R}^n$, this reduces to $\nabla f(x^*) = 0$



♠ If $\nabla f(x^*) \neq 0$, it defines supporting hyperplane to $\mathcal{X}$ at x^*

Optimality conditions – constrained

- Suppose $\exists y \in \mathcal{X}$ such that $\langle \nabla f(x^*), y - x^* \rangle < 0$
- Using mean-value theorem of calculus, $\exists \xi \in [0, 1]$ s.t.

$$f(x^* + t(y - x^*)) = f(x^*) + \langle \nabla f(x^* + \xi t(y - x^*)), t(y - x^*) \rangle$$

  (we applied MVT to $g(t) := f(x^* + t(y - x^*))$)
- For sufficiently small t, since ∇f continuous, by assump on y, $\langle \nabla f(x^* + \xi t(y - x^*)), y - x^* \rangle < 0$
- This in turn implies that $f(x^* + t(y - x^*)) < f(x^*)$
- Since $\mathcal{X}$ is convex, $x^* + t(y - x^*) \in \mathcal{X}$ is also feasible
- Contradiction to local optimality of x^*

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$$\operatorname{Argmin} f_0 = \mathsf{zer}(\partial f_0) := \{x \in \mathbb{R}^n \mid 0 \in \partial f_0(x)\}\,.$$

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Equivalently, $f_0(y) \geq f_0(x) + \langle 0,\, y - x\rangle \quad \forall y$, $\leftrightarrow 0 \in \partial f_0(x)$.

Nonsmooth optimality

$$\begin{aligned} \min \quad & f_0(x) \qquad \text{s.t. } x \in \mathcal{X} \\ \min \quad & f_0(x) + \mathbb{I}_{\mathcal{X}}(x). \end{aligned}$$

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♢ $-\nabla f(x^*) \in \mathcal{N}_{\mathcal{X}}(x^*) \iff \langle \nabla f(x^*),\, y - x^*\rangle \geq 0$ for all $y \in \mathcal{X}$.

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Observe: If constraint not satisfied strictly at optimum ($\|x\| < 1$), then $\nabla f(x) = 0$ (else we'd violate the last inequality above).

Equivalent Problems

Monotonic transformation

Standard form

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \leq 0, \quad 1 \leq i \leq m, \\ & Ax = b. \end{aligned}$$

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- Say $\psi_0 : \mathbb{R} \to \mathbb{R}$ is monotone increasing
- $\psi_i : \mathbb{R} \to \mathbb{R}$ satisfy $\psi_i(u) \leq 0$ iff $u \leq 0$
- $h(z) = 0$ iff $z = 0$.

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Transformed problem

$$\begin{aligned} \min \quad & \psi_0(f_0(x)) \\ \text{s.t.} \quad & \psi_i(f_i(x)) \le 0, \quad i = 1, \ldots, m \\ & h(Ax - b) = 0. \end{aligned}$$

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$\boxed{\text{Can destroy convexity}}$

Example

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- ♣ First problem is nondifferentiable
- ♣ Second is differentiable – solvable in closed form!

Slack variables

To turn inequalities into equalities

$$\min \quad f(x) \quad \text{s.t.} \quad Ax \leq b$$

$$\min_{x,s} \quad f(x) \quad \text{s.t.} \quad Ax + s = b, \; s \geq 0.$$

Epigraph form

Standard form; optimal value p^*

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At the optimum, $t = p^*$.

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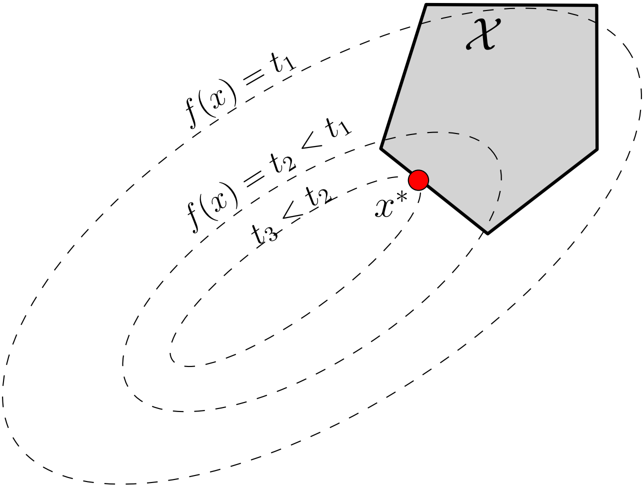
We seek smallest t such that L_t intersects with constraint set $\mathcal{X}$.

Epigraph form — geometrically



Variable elimination

$$\min_{x,y} \quad f_0(x,y) \quad \text{s.t.} \quad f_i(x) \le 0, \quad i = 1,\ldots,m.$$

Recall, since f_0 is convex in (x,y), $\inf_y f_0(x,y)$ is still convex.

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$\boxed{\text{Independent constraints important here.}}$

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Elimination form

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Introducing equality constraints

Separable function

$$\begin{array}{ll} \min & \sum_{i=1}^{T} f_{0,i}(x) \\ \text{s.t.} & f_i(x) \le 0, \quad i = 1, \ldots, m. \end{array}$$

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Often useful trick: **variable splitting**

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Almost separate problems! Useful for distributed computing.

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Penalized form (approximate)

$$\min \quad f_0(x) + \rho \|\max\{0, f(x)\}\|_2^2,$$

where $f(x) = [f_1(x), \ldots, f_m(x)]^T$; $\rho \gg 0$.

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Reducing number of constraints

$$\begin{aligned} &\min f_0(x) \quad \text{s.t.} \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\ \Longrightarrow &\min f_0(x) \quad \text{s.t.} \quad [g(x) := \max_{1 \leq i \leq m} f_i(x)] \leq 0. \end{aligned}$$

Implicit constraints

$$\min \quad c^T x - \sum_{i=1}^{m} \log(b_i - a_i^T x),$$

where $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and a_i^T are rows of $A \in \mathbb{R}^{m \times n}$.

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$\boxed{\text{Idea comes up again in interior point methods}}$

Problem classes

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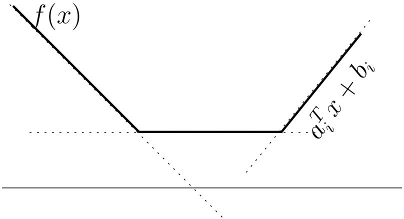

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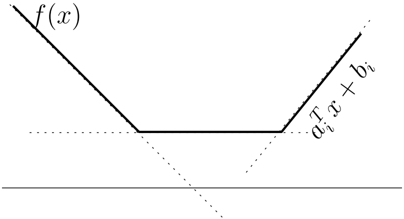

$$\min_{x,t} \quad t \qquad \text{s.t.} \quad a_i^T x + b_i \le t, \quad i = 1, \ldots, m.$$

- Linear program with variables $x, t \in \mathbb{R}$.

LP Exercises

- Formulate $\min \|Ax - b\|_1$ as an LP ($\|x\|_1 = \sum_i |x_|$)
- Formulate $\min \|Ax - b\|_\infty$ as an LP ($\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$)

Quadratic Programming

$$\min \quad \tfrac{1}{2}x^T Ax + b^T x + c \qquad \text{s.t.} \;\; Gx \leq h.$$

We assume $A \succeq 0$ (semidefinite).

Quadratic Programming

$$\min \quad \tfrac{1}{2}x^T Ax + b^T x + c \qquad \text{s.t.} \quad Gx \leq h.$$

We assume $A \succeq 0$ (semidefinite).

Exercise: Say no constraints; does this QP always have a solution?

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Nonnegative least squares (NNLS)

$$\min \quad \tfrac{1}{2}\|Ax - b\|^2 \quad \text{s.t. } x \geq 0.$$

Exercise: Prove that NNLS always has a solution.

Regularized least-squares

Lasso

$$\min \quad \frac{1}{2}\|Ax - b\|_2^2 + \lambda\|x\|_1.$$

Exercise: How large must $\lambda > 0$ so that $x = 0$ is the optimum?

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Total-variation denoising

$$\min \quad \tfrac{1}{2}\|Ax - b\|_2^2 + \lambda \sum_{i=1}^{n-1} |x_{i+1} - x_i|.$$

Exercise: Is the total-variation term a norm? Prove or disprove.

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Group Lasso

$$\min_{x_1,\ldots,x_T} \quad \frac{1}{2}\left\| b - \sum_{j=1}^{T} A_j x_j \right\|_2^2 + \lambda \sum_{j=1}^{T} \|x_j\|_2.$$

$\boxed{\text{Notice non-differentiable regularizers}}$

Second order cone program (SOCP)

$$\begin{aligned} \min \quad & f^T x \\ \text{s.t.} \quad & \|A_i x + b_i\|_2 \le c_i^T x + d_i, \quad i = 1, \ldots, m. \end{aligned}$$

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- Linear objective
- Nonlinear, nondifferentiable constraints
- Generalization of LP, QP: allows cone constraints
- Recall $\mathcal{Q}^n := \left\{(x, t) \in \mathbb{R}^{n+1} \mid \|x\|_2 \leq t\right\}$ is a convex cone

Example – robust LP

$$\min \quad c^T x, \qquad \text{s.t.} \;\; a_i^T x \leq b_i \; \forall a_i \in \mathcal{E}_i$$
$$\mathcal{E}_i := \{\bar{a}_i + P_i u \mid \|u\|_2 \leq 1\}$$

The constraints are uncertain but with bounded uncertainty.

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Semidefinite Program (SDP)

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^T x \\ \text{s.t.} \quad & A(x) := A_0 + x_1 A_1 + x_2 A_2 + \ldots + x_n A_n \succeq 0. \end{aligned}$$

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- When is a convex problem **representable** as an SDP?

Examples

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♠ Many more examples! See CVX documentation also.

♠ SDP relaxations of nonconvex problems — powerful, important

♠ More on this next lecture

References

1. L. Vandenberghe. MLSS 2012 Lecture slides.