Convex Optimization

(EE227A: UC Berkeley)

Lecture 3
(Convex sets and functions)

29 Jan, 2013

Suvrit Sra

Course organization

- http://people.kyb.tuebingen.mpg.de/suvrit/teach/ee227a/
- Relevant texts / references:
  - ♡ *Convex optimization* – Boyd & Vandenberghe (BV)
  - ♡ *Introductory lectures on convex optimisation* – Nesterov
  - ♡ *Nonlinear programming* – Bertsekas
  - ♡ *Convex Analysis* – Rockafellar
  - ♡ *Numerical optimization* – Nocedal & Wright
  - ♡ *Lectures on modern convex optimization* – Nemirovski
  - ♡ *Optimization for Machine Learning* – Sra, Nowozin, Wright
- Instructor: Suvrit Sra (suvrit@gmail.com)
  (Max Planck Institute for Intelligent Systems, Tübingen, Germany)
- HW + Quizzes (40%); Midterm (30%); Project (30%)
- TA Office hours to be posted soon
- I don't have an office yet ☺
- If you email me, please put **EE227A** in **Subject:**

Linear algebra recap

Eigenvalues and Eigenvectors

Def. If $A \in \mathbb{C}^{n\times n}$ and $x \in \mathbb{C}^n$. Consider the equation

$$Ax = \lambda x, \qquad x \neq 0, \quad \lambda \in \mathbb{C}.$$

If scalar λ and vector x satisfy this equation, then λ is called an **eigenvalue** and x and **eigenvector** of A.

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Eigenvalues are roots of characteristic polynomial.

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Theorem Let $\lambda_1, \ldots, \lambda_n$ be eigenvalues of $A \in \mathbb{C}^{n\times n}$. Then,

$$\mathrm{Tr}(A) = \sum_i a_{ii} = \sum_i \lambda_i, \qquad \det(A) = \prod_i \lambda_i.$$

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Theorem (Schur factorization). If $A \in \mathbb{C}^{n \times n}$ with eigenvalues $\lambda_1, \ldots, \lambda_n$, then there is a unitary matrix $U \in \mathbb{C}^{n \times n}$ (i.e., $U^*U = I$), such that

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Proof. $A = VTV^*$, $A^* = VT^*V^*$, so $AA^* = TT^* = T^*T = A^*A$. But T is upper triangular, so only way for $TT^* = T^*T$, some easy but tedious induction shows that T must be diagonal. Hence, $T = \Lambda$.

Singular value decomposition

Theorem (SVD) Let $A \in \mathbb{C}^{m\times n}$. There are unitaries s.t. U and V

$$U^*AV = \mathrm{Diag}(\sigma_1, \ldots, \sigma_p), \quad p = \min(m,n),$$

where $\sigma_1 \geq \sigma_2 \geq \cdots \sigma_p \geq 0$. Usually written as

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left singular vectors U are eigenvectors of AA^*

right singular vectors V are eigenvectors of A^*A

nonzero **singular values** $\sigma_i = \sqrt{\lambda_i(AA^*)} = \sqrt{\lambda_i(A^*A)}$

Positive definite matrices

Def. Let $A \in \mathbb{R}^{n \times n}$ be **symmetric**, i.e., $a_{ij} = a_{ji}$. Then, A is called **positive definite** if

$$x^T A x = \sum_{ij} x_i a_{ij} x_j > 0, \quad \forall\; x \neq 0.$$

If $>$ replaced by $\geq$, we call A **positive semidefinite**.

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Exercise: Prove this claim. Also prove converse.

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Notation: $A \succ 0$ (posdef) or $A \succeq 0$ (semidef)

Amongst most important objects in convex optimization!

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Easily derived using “brute-force” rules

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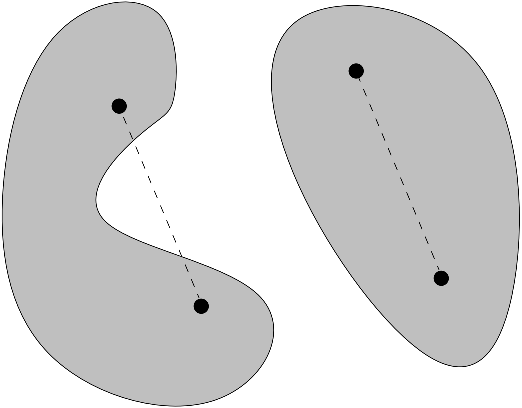
Easily derived using "brute-force" rules

- ♣ Wikipedia
- ♣ My ancient notes
- ♣ Matrix cookbook
- ♣ I hope to put up notes on less brute-forced approach.

Convex Sets

Convex sets



Convex sets

Def. A set $C \subset \mathbb{R}^n$ is called **convex**, if for any $x, y \in C$, the line-segment $\theta x + (1-\theta)y$ (here $\theta \geq 0$) also lies in C.

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- Convex: $\theta_1 x + \theta_2 y \in C$, where $\theta_1, \theta_2 \geq 0$ and $\theta_1 + \theta_2 = 1$.

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- Linear: if restrictions on θ_1, θ_2 are dropped
- Conic: if restriction $\theta_1 + \theta_2 = 1$ is dropped

Convex sets

Theorem (Intersection).
Let C_1, C_2 be convex sets. Then, $C_1 \cap C_2$ is also convex.

Proof. If $C_1 \cap C_2 = \emptyset$, then true vacuously.

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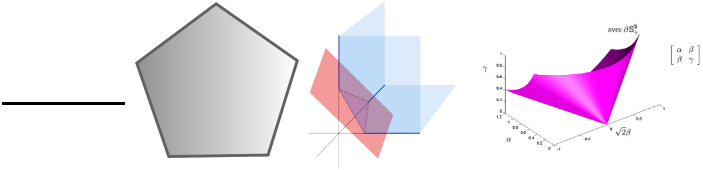
Thus, $\theta x + (1-\theta)y \in C_1 \cap C_2$.
Inductively follows that $\cap_{i=1}^{m} C_i$ is also convex.

Convex sets – more examples



(psdcone image from convexoptimization.com, Dattorro)

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———————— ∘ ————————

Quiz: Prove that these sets are convex.

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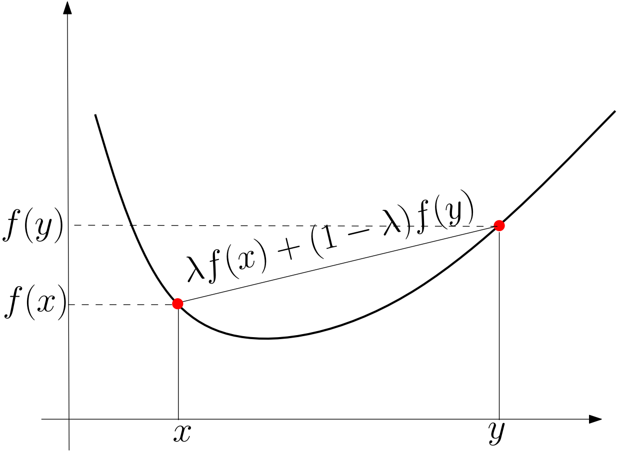
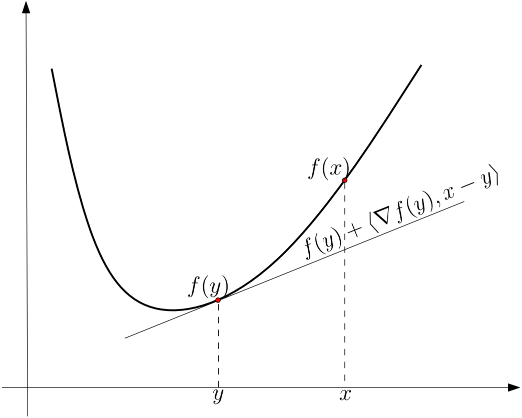
▶ Theorem extends to functions $f : \mathcal{X} \subseteq \mathbb{R}^n \to \mathbb{R}$. Very useful to checking convexity of a given function.

Convex functions



$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

Convex functions



$$f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle$$

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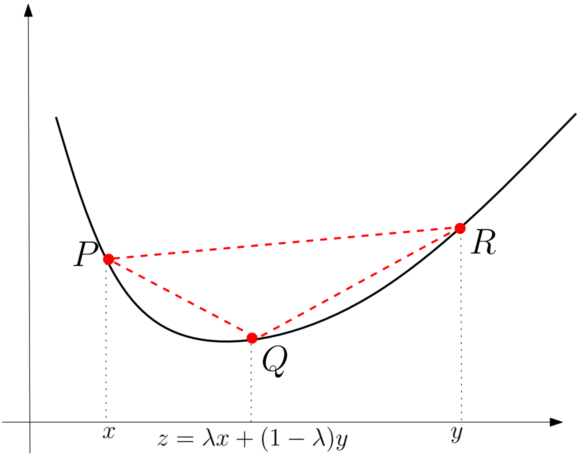

slope PQ $\leq$ slope PR $\leq$ slope QR

Recognizing convex functions

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- ♠ If f is twice differentiable, then f is convex *if and only if* $\operatorname{dom} f$ is convex and $\nabla^2 f(x) \succeq 0$ at every $x \in \operatorname{dom} f$.

Convex functions

- Linear: $f(\theta_1 x + \theta_2 y) = \theta_1 f(x) + \theta_2 f(y)$; θ_1, θ_2 unrestricted
- Concave: $f(\theta x + (1-\theta)y) \geq \theta f(x) + (1-\theta) f(y)$
- Strictly convex: If inequality is strict for $x \neq y$

Convex functions

Example The *pointwise maximum* of a family of convex functions is convex. That is, if $f(x; y)$ is a convex function of x for every y in some "index set" $\mathcal{Y}$, then

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Exercise: Verify truth of above examples.

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Theorem Let $\mathcal{Y}$ be a nonempty convex set. Suppose $L(x, y)$ is convex in (x, y), then,

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Since $\epsilon > 0$ is arbitrary, claim follows.

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Let A, B, C be matrices such that $C \succ 0$, and let

$$Z := \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succeq 0,$$

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(We skipped ahead and solved $\nabla_y L(x, y) = 0$ to minimize).

Recognizing convex functions

- If f is continuous and midpoint convex, then it is convex.
- If f is differentiable, then f is convex *if and only if* $\operatorname{dom} f$ is convex and $f(x) \geq f(y) + \langle \nabla f(y),\, x - y \rangle$ for all $x, y \in \operatorname{dom} f$.
- If f is twice differentiable, then f is convex *if and only if* $\operatorname{dom} f$ is convex and $\nabla^2 f(x) \succeq 0$ at every $x \in \operatorname{dom} f$.

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- By showing $f : \text{dom}(f) \to \mathbb{R}$ is convex *if and only if* its restriction to **any** line that intersects $\operatorname{dom}(f)$ is convex. That is, for any $x \in \operatorname{dom}(f)$ and any v, the function $g(t) = f(x + tv)$ is convex (on its domain $\{t \mid x + tv \in \operatorname{dom}(f)\}$).
- See exercises (Ch. 3) in Boyd & Vandenberghe for more ways

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Affine composition: $f(x) := g(Ax + b)$, where g is convex.

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Theorem Let $f : I_1 \to \mathbb{R}$ and $g : I_2 \to \mathbb{R}$, where $\text{range}(f) \subseteq I_2$. If f and g are convex, and g is increasing, then $g \circ f$ is convex on I_1

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Read Section 3.2.4 of BV for more

Examples

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Let $f(x) = x^T Ax + b^T x + c$, where $A \succeq 0$, $b \in \mathbb{R}^n$, and $c \in \mathbb{R}$.

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What is: $\nabla^2 f(x)$?

$\nabla f(x) = 2Ax + b$, $\nabla^2 f(x) = A \succeq 0$, hence f is convex.

Indicator

Let $\mathbb{I}_{\mathcal{X}}$ be the *indicator function* for $\mathcal{X}$ defined as:

$$\mathbb{I}_{\mathcal{X}}(x) := \begin{cases} 0 & \text{if } x \in \mathcal{X}, \\ \infty & \text{otherwise.} \end{cases}$$

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Note: $\mathbb{I}_{\mathcal{X}}(x)$ is convex **if and only if** $\mathcal{X}$ is convex.

Distance to a set

Example Let $\mathcal{Y}$ be a convex set. Let $x \in \mathbb{R}^n$ be some point. The distance of x to the set $\mathcal{Y}$ is defined as

$$\text{dist}(x, \mathcal{Y}) := \inf_{y \in \mathcal{Y}} \quad \|x - y\|.$$

Because $\|x - y\|$ is jointly convex in (x, y), the function $\text{dist}(x, \mathcal{Y})$ is a convex function of x.

Norms

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function that satisfies

1. $f(x) \geq 0$, and $f(x) = 0$ if and only if $x = 0$ (definiteness)
2. $f(\lambda x) = |\lambda| f(x)$ for any $\lambda \in \mathbb{R}$ (positive homogeneity)
3. $f(x + y) \leq f(x) + f(y)$ (subadditivity)

Such a function is called a *norm*. We usually denote norms by $\|x\|$.

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Theorem Norms are convex.

Proof. Immediate from subadditivity and positive homogeneity.

Vector norms

Example (ℓ_2-norm): Let $x \in \mathbb{R}^n$. The **Euclidean** or ℓ_2-norm is $\|x\|_2 = \left(\sum_i x_i^2\right)^{1/2}$

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Example (Frobenius-norm): Let $A \in \mathbb{R}^{m \times n}$. The **Frobenius** norm of A is $\|A\|_{\mathsf{F}} := \sqrt{\sum_{ij} |a_{ij}|^2}$; that is, $\|A\|_{\mathsf{F}} = \sqrt{\mathrm{Tr}(A^*A)}$.

Mixed norms

Def. Let $x \in \mathbb{R}^{n_1+n_2+\cdots+n_G}$ be a vector partitioned into subvectors $x_j \in \mathbb{R}^{n_j}$, $1 \leq j \leq G$. Let $\boldsymbol{p} := (p_0, p_1, p_2, \ldots, p_G)$, where $p_j \geq 1$. Consider the vector $\xi := (\|x_1\|_{p_1}, \cdots, \|x_G\|_{p_G})$. Then, we define the **mixed-norm** of x as

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Example $\ell_{1,q}$-norm: Let x be as above.

$$\|x\|_{1,q} := \sum_{i=1}^{G} \|x_i\|_q.$$

This norm is popular in machine learning, statistics.

Matrix Norms

Induced norm

Let $A \in \mathbb{R}^{m \times n}$, and let $\|\cdot\|$ be any vector norm. We define an **induced matrix norm** as

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- Clearly, $\|A\| = 0$ iff $A = 0$ (definiteness)
- $\|\alpha A\| = |\alpha| \; \|A\|$ (homogeneity)
- $\|A + B\| = \sup \frac{\|(A+B)x\|}{\|x\|} \leq \sup \frac{\|Ax\|+\|Bx\|}{\|x\|} \leq \|A\| + \|B\|.$

Operator norm

Example Let A be any matrix. Then, the **operator norm** of A is

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$\|A\|_2 = \sigma_{\max}(A)$, where $\sigma_{\max}$ is the largest singular value of A.

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- $\|A\|_p$ generally NP-Hard to compute for $p \notin \{1, 2, \infty\}$

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- $\|A\|_p$ generally NP-Hard to compute for $p \notin \{1, 2, \infty\}$
- **Schatten p-norm:** ℓ_p-norm of vector of singular value.

Operator norm

Example Let A be any matrix. Then, the **operator norm** of A is

$$\|A\|_2 := \sup_{\|x\|_2 \neq 0} \frac{\|Ax\|_2}{\|x\|_2}.$$

$\|A\|_2 = \sigma_{\max}(A)$, where $\sigma_{\max}$ is the largest singular value of A.

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- **Schatten p-norm:** ℓ_p-norm of vector of singular value.
- **Exercise:** Let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$ be singular values of a matrix $A \in \mathbb{R}^{m \times n}$. Prove that

$$\|A\|_{(k)} := \sum_{i=1}^{k} \sigma_i(A),$$

is a norm; $1 \leq k \leq n$.

Dual norms

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Exercise: Verify that $\|u\|_*$ is a norm.

Exercise: Let $1/p + 1/q = 1$, where $p, q \geq 1$. Show that $\|\cdot\|_q$ is dual to $\|\cdot\|_p$. In particular, the ℓ_2-norm is self-dual.

Fenchel Conjugate

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Example $f(x) = \max(0, 1 - x)$. Now $f^*(z) = \sup_x zx - \max(0, 1 - x)$. Note that $\operatorname{dom} f^*$ is $[-1, 0]$ (else sup is unbounded); within this domain, $f^*(z) = z$.

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- ♣ **Discrete convexity:** $f : \mathbb{Z}^n \to \mathbb{Z}$; "convexity + matroid theory."