Convex Optimization

(EE227A: UC Berkeley)

Lecture 26
Interior point methods

25 Apr, 2013

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Interior point methods

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- Interior Point Methods build on the Newton method to ultimately tackle the above convex optimization problem

Preliminaries

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- Let **central path** be $\{x^*(t) \mid t \geq 0\}$; as $t \to \infty$, central path converges to solution of original problem.

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    1. Replace penalty t_k by a larger value t_{k+1}
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- Any unconstrained method to solve for x_{k+1}
- What is complexity of such a scheme?
- Numerical problems when t_k becomes large; breakdown?
- Standard theory of unconstrained minimization predicts slowdown as t_k becomes larger ...

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> Shortly thereafter, Nesterov realized what intrinsic properties of the log-barrier made it work!

Newton method – affine invariance

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Lemma Let $\{x_k\}$ be generated by Newton method for f:

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Let $\{y_k\}$ be seq. generated by NM for ϕ:

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Newton method remains same—strong contrast to gradient method!

Stopping condition:

$$\langle [f''(x_k)]^{-1} f'(x_k),\, f'(x_k) \rangle < \epsilon$$

independent of choice of basis A!

Newton method – local convergence

Assumptions

- **Lipschitz Hessian**: $\|\nabla^2 f(x) - \nabla^2 f(y)\| \leq M\|x - y\|$
- **Local strong convexity**: There exists a local minimum x^* with

$$\nabla^2 f(x^*) \succeq \mu I, \qquad \mu > 0.$$

- **Locality**: Starting point x_0 "close enough" to x^*

Theorem Suppose x_0 satisfies

$$\|x_0 - x^*\| < r := \frac{2\mu}{3M}.$$

Then, $\|x_k - x^*\| < r$, $\forall k$ and the NM converges quadratically

$$\|x_{k+1} - x^*\| \leq \frac{M\|x_k - x^*\|^2}{2(\mu - M\|x_k - x^*\|)}$$

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- Mismatch between geometry of method and its convergence analysis

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- ☞ Thus, at $x \in \operatorname{dom} f$, and any $u, v \in \mathbb{R}^n$ we have

$$\langle f'''(x)[u]v,\, v\rangle \le M\|u\|\,\|v\|^2$$

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☞ Using $x \leftarrow Ay$, $u' \leftarrow Au$, $v' \leftarrow Av$, $\phi(y) = f(Ay)$

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- This brings us to the idea of **self-concordance**

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- Denote restriction to line $\phi(x;t) := f(x+tu)$

Derivatives

$$\begin{aligned} Df(x)[u] &= \phi'(x;t) = \langle f'(x),\, u\rangle \\ D^2f(x)[u,u] &= \phi''(x;t) = \langle f''(x)u,\, u\rangle = \|u\|^2_{f''(x)} \\ D^3f(x)[u,u,u] &= \phi'''(x;t) = \langle D^3f(x)[u]u,\, u\rangle \end{aligned}$$

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$$

Note: Third derivative: symmetric trilinear operator, so it operates on $[u_1,u_2,u_3]$ to yield a trilinear symmetric form.

$$
D^p f(x)[u_1,\ldots,u_p] = \left.\frac{\partial^p}{\partial t_1 \cdots \partial t_p}\right|_{t_1=\cdots=t_p=0} f(x+t_1u_1+\cdots+t_pu_p)
$$

Self-concordant functions and barriers

Def. (**Self-concordant**). Let $\mathcal{X}$ be a closed convex set. A function $f : \mathrm{int}(\mathcal{X}) \to \mathbb{R}$ called **self-concordant** (SC) on $\mathcal{X}$ if

- ☞ $f \in C^3(\mathcal{X})$ with $f(x_k) \to +\infty$ if $x_k \to \bar{x} \in \partial\mathcal{X}$
- ☞ f satisfies the **SC inequality**

$$|D^3 f(x)[u,u,u]| \le 2\left(D^2 f(x)[u,u]\right)^{3/2}, \quad \forall x \in \mathrm{int}(\mathcal{X}), u \in \mathbb{R}^n$$

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Def. Given a real $\vartheta \ge 1$, F is called a **ϑ-self-concordant barrier** (SCB) for $\mathcal{X}$ if F is SC and

$$|DF(x)[u]| \le \vartheta^{1/2}\left(D^2 f(x)[u,u]\right)^{1/2}, \quad \forall x \in \text{int}(\mathcal{X}), u \in \mathbb{R}^n.$$

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- Exponents $3/2$ and $1/2$ crucial—ensure both sides have same degree of homogeneity in u (for affine invariance)
- Factor 2 can be scaled by scaling f; chosen for convenience; equiv. to $D^2 f$ Lipschitz with constant 2 in norm $\|\cdot\|_{f''(x)}$

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- Linear functions are SC; $f'''(x) = 0$
- Convex quadratic functions; $f'''(x) = 0$
- Log-barrier for $\phi(x) = a + \langle b,\, x\rangle - \frac{1}{2}x^T Ax$; $f(x) = -\log\phi(x)$
  Show: $|D^3 f(x)[u,u,u]| = |2\omega_1^3 + 3\omega_1\omega_2|$, where $\omega_1 = Df(x)[u]$, $\omega_2 = \frac{1}{\phi(x)}u^T Au$; also show that $D^2 f(x)[u,u] = \omega_1^2 + \omega_2$.

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Lemma A function f is SC iff for any $x \in \text{int}(\mathcal{X})$, and $u_1, u_2, u_3 \in \mathbb{R}^n$

$$|D^3 f(x)[u_1,u_2,u_3]| \le 2\|u_1\|_{f''(x)}\|u_1\|_{f''(x)}\|u_1\|_{f''(x)}$$

Proof: Essentially generalized Cauchy-Schwarz (some work).

SC Optimization

Key quantities

- Let $f(x)$ be SC, and that $f''(x) \succ 0$ within $\operatorname{dom} f$
- Simplified notation for the local norms at x

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- Let $f(x)$ be SC, and that $f''(x) \succ 0$ within $\operatorname{dom} f$
- Simplified notation for the local norms at x

$$\begin{aligned} \|u\|_x &:= \langle f''(x)u,\, u\rangle^{1/2} \\ \|v\|_x^* &= \langle [f''(x)]^{-1}v,\, v\rangle^{1/2} \end{aligned}$$

- Let us use these to state three crucial observations

Three key facts

☞ At any point $x \in \operatorname{dom} f = \text{int}(\mathcal{X})$, there is an **ellipsoid**

$$W(x) := \{y \in \mathbb{R}^n \mid \|y - x\|_x \leq 1\} \subset \operatorname{dom} f.$$

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☞ Moreover, it also holds that

$$f(x) + \langle f'(x),\, u\rangle + \rho(-r) \le f(x+u) \le f(x) + \langle f'(x),\, u\rangle + \rho(r),$$

where $\rho(r) := -\log(1-r) - s = s^2/2 + s^3/3 + \cdots$

Proof: See Chap. 4 of Nesterov (2004).

Setting up Newton's method

Newton decrement

$$\lambda_f(x) := \langle [f''(x)]^{-1} f'(x),\ f'(x) \rangle^{1/2}.$$

Observe: $\lambda_f(x) = \|f'(x)\|_x^*$ (local, dual-norm of gradient).

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Theorem If $\lambda_f(x) < 1$ for some $x \in \operatorname{dom} f$. Then, $\min f(x)$ s.t., $x \in \operatorname{dom} f$, has a unique optimal solution.

Damped Newton method

1. Select $x_0 \in \operatorname{dom} f$
2. For $k \geq 0$: $x_{k+1} = x_k - \frac{1}{1+\lambda_f(x_k)}[f''(x_k)]^{-1} f'(x_k)$

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At each step, $f(x)$ decreases by at least $\omega(\lambda)$

Damped Newton method

• Globally convegent; iteration complexity can be derived.

• Local quadratic convergence: $\lambda_f(x_{k+1}) \leq 2\lambda_f(x_k)^2$ for small enough $\lambda_f(x_k)$

• Though, better to start with DN, and switch to pure Newton after N iterations, where

$$N \leq \frac{1}{\omega(\beta)[f(x_0) - f(x_f^*)]},$$

and $\lambda_f(x_k) \geq \beta$, where $\beta \in (0, 0.3819...)$

Minimization using SC Barriers

- class of ϑ-SCB smaller than general SC.

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Standard convex problem

$$\min \quad c^T x \quad x \in \mathcal{X},$$

where $\mathcal{X}$ is a compact set for which $\operatorname{dom} F \equiv \mathcal{X}$.

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- Recall path-following scheme

$$x^*(t) = \underset{x \in \operatorname{dom} F}{\operatorname{argmin}} \quad tc^T x + F(x), \quad t \geq 0.$$

- Any point of the central path (set $\{x^*(t)\}$) satisfies

$$tc + F'(x^*(t)) = 0.$$

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- Aim is to iteratively find points close to central path

Minimization using SCBs

Approximate solution:

$$\lambda_{F_t}(x) := \|F_t'(x)\|_x^* = \|tc + F'(x)\|_x^* \le \beta,$$

where β is the centering parameter (approx. solution quality).

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Theorem For any $t > 0$, we have

$$c^T x^*(t) - c^T x^* \le \frac{\vartheta}{t}.$$

If a point x is an approximate solution (close to $x^*(t)$), then

$$c^T x - c^T x^* \le \frac{1}{t}\left(\vartheta + \frac{\beta(\beta + \sqrt{\vartheta})}{1 - \beta}\right).$$

Path-following algorithm

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Theorem Above scheme yields $c^T x_N - c^T x^* \le \epsilon$ after no more than N steps, where
$$N \le O\left(\sqrt{\vartheta} \log \frac{\vartheta \|c\|_{x^*}^*}{\epsilon}\right).$$

More

We've barely scratched the surface!

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- Much more to interior point methods.
- See references for fuller picture.

Also read: Ch. 9,10,11 of BV for high-level overview.

References

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