Convex Optimization

(EE227A: UC Berkeley)

Lecture 25
(Newton, quasi-Newton)

23 Apr, 2013

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Admin

- Project **poster** presentations:

Soda 306 HP Auditorium Fri May 10, 2013 4pm – 8pm

- HW5 due on **May 02, 2013**
  Will be released today.

Newton method

- Recall numerical analysis: Newton method for solving equations

$$g(x) = 0 \quad x \in \mathbb{R}.$$

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- Which suggests the iterative process

$$x_{k+1} \leftarrow x_k - \frac{g(x_k)}{g'(x_k)}$$

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where $G'(x)$ is the **Jacobian**.

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- Assume $G'(x)$ is non-degenerate (invertible), we obtain

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- This is Newton's method for solving nonlinear equations

Newton method

$$\boxed{\min \quad f(x) \text{ such that } x \in \mathbb{R}^n}$$

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Newton system

$$\nabla f(x) + \nabla^2 f(x) \Delta x = 0,$$

which leads to

$$x_{k+1} = x_k - [\nabla^2 f(x_k)]^{-1} \nabla f(x_k).$$

the **Newton method** for optimization

Newton method – remarks

- Newton method for equations is **more general** than minimizing $f(x)$ by finding roots of $\nabla f(x) = 0$

Newton method – remarks

- Newton method for equations is **more general** than minimizing $f(x)$ by finding roots of $\nabla f(x) = 0$
- **Reason:** Not every function $G : \mathbb{R}^n \to \mathbb{R}^n$ is a derivative!

Example Consider the linear system

$$Ax - b = 0.$$

Unless A is symmetric, does not correspond to a derivative (Why?)

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- If it were a derivative, then its own derivative is a Hessian, and we know that Hessians must be symmetric, QED.

Newton method – remarks

- In general, Newton method highly nontrivial to analyze

Example Consider the iteration

$$x_{k+1} = x_k - \frac{1}{x_k}, \qquad x_0 = 2.$$

May be viewed as iter for $e^{x^2/2} = 0$ (which has **no real solution**)

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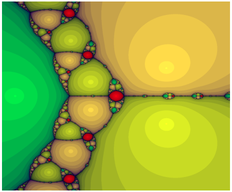
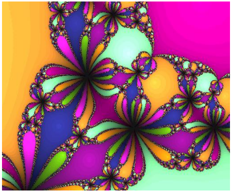
Newton fractals (Complex dynamics)



$z^3 - 2z + 2$



$x^8 + 15x^4 - 16$

Newton method – alternative view

Quadratic approximation

$$\phi(x) := f(x) + \langle \nabla f(x_k),\, x - x_k \rangle + \tfrac{1}{2}\langle \nabla^2 f(x_k)(x - x_k),\, x - x_k \rangle.$$

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Assuming $\nabla^2 f(x_k) \succ 0$, choose x_{k+1} as argmin of $\phi(x)$

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Assuming $\nabla^2 f(x_k) \succ 0$, choose x_{k+1} as argmin of $\phi(x)$

$$\phi'(x_{k+1}) = \nabla f(x_k) + \nabla^2 f(x_k)(x_{k+1} - x_k) = 0.$$

Newton method – convergence

- Method breaks down if $\nabla^2 f(x_k) \not\succ 0$
- Only **locally convergent**

Example Find the root of

$$g(x) = \frac{x}{\sqrt{1+x^2}}.$$

Clearly, $x^* = 0$.

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Exercise: Analyze behavior of Newton method for this problem.
Hint: Consider the cases: $|x_0| < 1$, $x_0 = \pm 1$ and $|x_0| > 1$.

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Damped Newton method

$$\boxed{x_{k+1} = x_k - \alpha_k [\nabla^2 f(x_k)]^{-1} \nabla f(x_k)}$$

Newton – local convergence rate

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- Let $g(x_k) \equiv \nabla f(x_k)$; Taylor's theorem:

$$0 = g(x^*) = g(x_k) + \langle \nabla g(x_k),\, x^* - x_k \rangle + o(\|x_k - x^*\|)$$

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- Multiply by $[\nabla g(x_k)]^{-1}$ to obtain

$$x_k - x^* - [\nabla g(x_k)]^{-1} g(x_k) = o(\|x_k - x^*\|)$$

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$$x_{k+1} - x^* = o(\|x_k - x^*\|),$$
- So for $x_k \neq x^*$ we get
$$\lim_{k\to\infty} \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|} = \lim_{k\to\infty} \frac{o(\|x_{k+1} - x^*\|)}{\|x_k - x^*\|} = 0.$$

Local superlinear convergence rate

Newton method – local convergence

Assumptions

- **Lipschitz Hessian**: $\|\nabla^2 f(x) - \nabla^2 f(y)\| \leq M\|x - y\|$
- **Local strong convexity**: There exists a local minimum x^* with

$$\nabla^2 f(x^*) \succeq \mu I, \qquad \mu > 0.$$

- **Locality**: Starting point x_0 "close enough" to x^*

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Theorem Suppose x_0 satisfies

$$\|x_0 - x^*\| < r := \frac{2\mu}{3M}.$$

Then, $\|x_k - x^*\| < r$, $\forall k$ and the NM converges quadratically

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Reading assignment: Read §9.5.3 of Boyd-Vandenberghe

Quasi-Newton

Gradient and Newton

$$(\text{Grad}) \qquad x_{k+1} = x_k - \alpha_k \nabla f(x_k), \quad \alpha_k > 0$$

$$(\text{Newton}) \qquad x_{k+1} = x_k - [\nabla^2 f(x_k)]^{-1} \nabla f(x_k).$$

Gradient and Newton

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Viewpoint for the **gradient method**.

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$$\phi_1(x) := f(x_k) + \langle \nabla f(x_k), x - x_k \rangle + \frac{1}{2\alpha}\|x - x_k\|^2$$

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Optimality condition yields

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$$\begin{aligned} \phi'(x^*) &= \nabla f(x_k) + \tfrac{1}{\alpha}(x^* - x_k) = 0 \\ x^* &= x_k - \alpha \nabla f(x_k) \end{aligned}$$

If $\alpha \in (0, \frac{1}{L}]$, $\phi_1(x)$ is **global overestimator**

$$f(x) \leq \phi_1(x), \qquad \forall x \in \mathbb{R}^n.$$

Gradient and Newton

Viewpoint for **Newton method**. Consider quadratic approx

$$\phi_2(x) := f(x_k) + \langle \nabla f(x_k),\, x - x_k\rangle + \frac{1}{2}\langle \nabla^2 f(x_k)(x - x_k),\, x - x_k\rangle.$$

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Something better than ϕ_1, less expensive than ϕ_2?

Quasi-Newton methods

Generic Quadratic Model

$$\phi_D(x) := f(x_k) + \langle \nabla f(x_k),\, x - x_k \rangle + \tfrac{1}{2}\langle H_k(x - x_k),\, x - x_k \rangle.$$

Quasi-Newton methods

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- Matrix $H_k \succ 0$, some posdef matrix
- Leads to optimum

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- The first-order methods that form a sequence of matrices

$$\{H_k\} \;:\; H_k \to \nabla^2 f(x^*)$$

where H_k is constructed using **only gradient information**,

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- The first-order methods that form a sequence of matrices

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where H_k is constructed using **only gradient information**,are called **variable metric** or **quasi-Newton** methods.

$$x_{k+1} = x_k - H_k^{-1}\nabla f(x_k) \qquad k = 0, 1, \ldots$$
$$x_{k+1} = x_k - S_k \nabla f(x_k) \qquad k = 0, 1, \ldots$$

Quasi-Newton method

- Choose $x_0 \in \mathbb{R}^n$. Let $H_0 = I$.
  Compute $f(x_0)$ and $\nabla f(x_0)$

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  5. **QN update:** $S_k \to S_{k+1}$

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  4. compute $f(x_{k+1})$ and $\nabla f(x_{k+1})$
  5. **QN update:** $S_k \to S_{k+1}$

QN schemes differ in how $S_k \equiv H_k^{-1}$ are updated!

Quasi-Newton methods

Secant equation / QN rule

$$S_{k+1}(\nabla f(x_{k+1}) - \nabla f(x_k)) = x_{k+1} - x_k.$$

Quasi-Newton methods

Secant equation / QN rule

$$S_{k+1}(\nabla f(x_{k+1}) - \nabla f(x_k)) = x_{k+1} - x_k.$$

- Quadratic models from iteration $k \to k+1$

$$\phi_k(x) = a_k + \langle g_k,\, x - x_k\rangle + \tfrac{1}{2}\langle H(x - x_k),\, x - x_k\rangle$$
$$\phi_{k+1}(x) = a_{k+1} + \langle g_{k+1},\, x - x_{k+1}\rangle + \tfrac{1}{2}\langle H(x - x_{k+1}),\, x - x_{k+1}\rangle$$

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- Quadratic models from iteration $k \to k+1$

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- $\phi_k'(x) - \phi_{k+1}'(x) = g_k - g_{k+1} + H(x_{k+1} - x_k)$
- Setting this to zero, we get

$$\begin{aligned} g_{k+1} - g_k &= H(x_{k+1} - x_k) \\ S(g_{k+1} - g_k) &= x_{k+1} - x_k. \end{aligned}$$

- So we construct $H_k \to H_{k+1}$ or $S_k \to S_{k+1}$ to respect this.

Hessian updates

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- Notice, updates computationally "cheap"

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$$\boxed{\text{Hessian storage and update has } O(n^2) \text{ cost}}$$

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Hessian storage and update has $O(n^2)$ cost

Estimate H_k or S_k using only previous few iterations; so essentially, use only $O(mn)$ storage, where $m \approx$ 5-17

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- We use m vector pairs (s_i, y_i), for $i = k-m, \ldots, k-1$

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- This is related to the BB stepsize!

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Two-metric projection method

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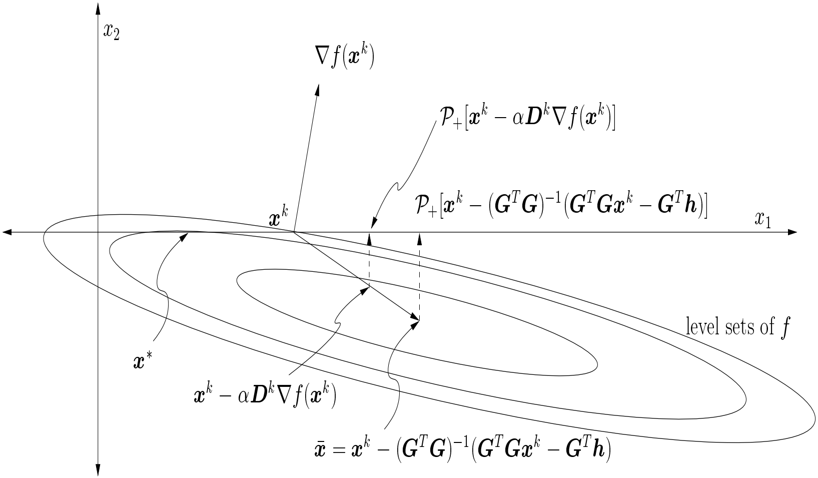
- Method might not even **recognize a stationary point!**

Failure of projected-Newton methods



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- With simple bound constraints: LBFGS-B

Nonsmooth problems

We did not cover many interesting ideas

- ♠ Proximal Newton methods
- ♠ $f(x) + r(x)$ problems (see book chapter)
- ♠ Nonsmooth BFGS – Lewis, Overton
- ♠ Nonsmooth LBFGS

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