Convex Optimization

(EE227A: UC Berkeley)

Lecture 12
(Subgradient methods)

28 Feb, 2013

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Announcements

- Midterm: final date is 19th March

ALGORITHMS

Unconstrained convex problem

$$\min_x \quad f(x)$$

How to optimize?

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- In reality: we stop in finite time
- Only solve problem approximately
- $f(x^k) \leq f(x^*) + \varepsilon$
- **shorthand** $f^k \leq f^* + \varepsilon$

Subgradient method

$$x^{k+1} = x^k - \alpha_k g^k$$

where $g^k \in \partial f(x^k)$ is **any** subgradient

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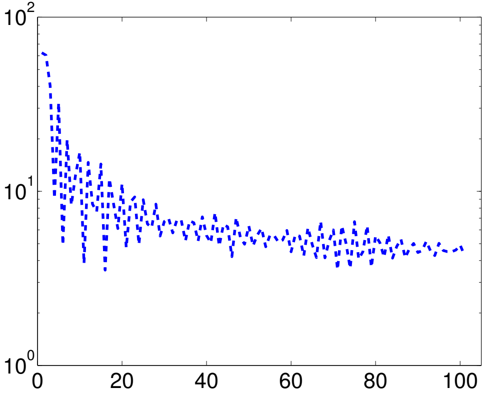
Stepsize $\alpha_k > 0$ must be chosen

- Method generates sequence $\{x^k\}_{k\geq 0}$
- Does this sequence converge to an optimal solution x^*?
- If yes, then how fast?
- What if have constraints: $x \in \mathcal{C}$?

Example

$$\min \quad \frac{1}{2}\|Ax-b\|_2^2+\lambda\|x\|_1$$
$$x^{k+1}=x^k-\alpha_k(A^T(Ax^k-b)+\lambda\,\mathrm{sgn}(x^k))$$



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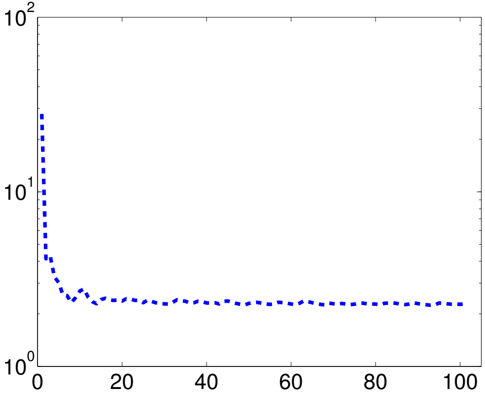

(More careful implementation)

Subgradient method – stepsizes

- **Constant** Set $\alpha_k = \alpha > 0$, for $k \geq 0$
- **Scaled constant** $\alpha_k = \alpha/\|g^k\|_2 \quad (\|x^{k+1} - x^k\|_2 = \alpha)$

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- **Scaled constant** $\alpha_k = \alpha/\|g^k\|_2 \quad (\|x^{k+1} - x^k\|_2 = \alpha)$
- **Square summable but not summable**

$$\sum_k \alpha_k^2 < \infty, \qquad \sum_k \alpha_k = \infty$$

- **Diminishing scalar**

$$\lim_k \alpha_k = 0, \qquad \sum_k \alpha_k = \infty$$

- **Adaptive stepsizes** (not covered)

Not a descent method!
Work with best f^k so far: $f^k_{\min} := \min_{0 \leq i \leq k} f^i$

Convergence analysis

Assumptions

- Min is attained: $f^* := \inf_x f(x) > -\infty$, with $f(x^*) = f^*$

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  ($f(x) - f(y) = \langle g_\xi, x - y \rangle$; use Cauchy-Schwarz or Hölder)

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Convergence results for: $f^k_{\min} := \min_{0 \le i \le k} f^i$

Subgradient method – convergence

Lyapunov function: **Distance to x^***, not function values

Subgradient method – convergence

Lyapunov function: Distance to x^*, not function values

$$\|x^{k+1} - x^*\|_2^2 \quad = \quad \|x^k - \alpha_k g^k - x^*\|_2^2$$

Subgradient method – convergence

Lyapunov function: **Distance to x^***, not function values

$$
\begin{aligned}
\|x^{k+1} - x^*\|_2^2 &= \|x^k - \alpha_k g^k - x^*\|_2^2 \\
&= \|x^k - x^*\|_2^2 + \alpha_k^2 \|g^k\|_2^2 - 2\langle \alpha_k g^k, x^k - x^* \rangle
\end{aligned}
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&\leq \|x^k - x^*\|_2^2 + \alpha_k^2 \|g^k\|_2^2 - 2\alpha_k (f(x^k) - f^*),
\end{aligned}
$$

since $f^* = f(x^*) \geq f(x^k) + \langle g^k, x^* - x^k \rangle$

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since $f^* = f(x^*) \geq f(x^k) + \langle g^k, x^* - x^k \rangle$

Apply same argument to $\|x^k - x^*\|_2^2$ recursively

Subgradient method – convergence

Lyapunov function: Distance to x^*, not function values

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Apply same argument to $\|x^k - x^*\|_2^2$ recursively

$$\|x^{k+1} - x^*\|_2^2 \leq \|x^0 - x^*\|_2^2 + \sum_{t=1}^{k} \alpha_t^2 \|g^k\|_2^2 - 2 \sum_{t=1}^{k} \alpha_t (f^t - f^*).$$

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Now use our convenient assumptions!

Subgradient method – convergence

$$\|x^{k+1} - x^*\|_2^2 \leq R^2 + G^2 \sum\nolimits_{t=1}^{k} \alpha_t^2 - 2 \sum\nolimits_{t=1}^{k} \alpha_t (f^t - f^*).$$

- To get a bound on the last term, simply notice (for $t \leq k$)

$$f^t \geq f_{\min}^t \geq f_{\min}^k \qquad \text{since} \quad f_{\min}^t := \min_{0 \leq i \leq t} f(x^i)$$

Subgradient method – convergence

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- Plugging this in yields the bound

$$2 \sum_{t=1}^{k} \alpha_t (f^t - f^*) \geq 2(f^k_{\min} - f^*) \sum_{t=1}^{k} \alpha_t.$$

Subgradient method – convergence

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- So that we finally have

$$0 \leq \|x^{k+1} - x^*\|_2 \leq R^2 + G^2 \sum_{t=1}^{k} \alpha_t^2 - 2(f_{\min}^k - f^*) \sum_{t=1}^{k} \alpha_t$$

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$$\boxed{f^k_{\min} - f^* \leq \frac{R^2 + G^2 \sum_{t=1}^{k} \alpha_t^2}{2 \sum_{t=1}^{k} \alpha_t}}$$

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Exercise: Analyze $\lim_{k \to \infty} f^k_{\min} - f^*$ for the different choices of stepsize that we mentioned.

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Constant step: $\alpha_k = \alpha$; We obtain

$$f_{\min}^k - f^* \leq \frac{R^2 + G^2 k\alpha^2}{2k\alpha}$$

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As $k \to \infty$, numerator $< \infty$ but denominator $\to \infty$; so $f_{\min}^k \to f^*$

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In practice, fair bit of stepsize tuning needed, e.g. $\alpha_t = a/(b+t)$

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- For fixed k: best possible stepsize is constant α

$$\frac{R^2 + G^2 k \alpha^2}{2k\alpha} \leq \epsilon \quad \Rightarrow \quad \alpha = \frac{R}{G\sqrt{k}}$$

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- Then, after k steps $f_{\min}^k - f^* \le RG/\sqrt{k}$.
- For accuracy ϵ, we need at least $(RG/\epsilon)^2 = O(1/\epsilon^2)$ steps

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- For accuracy ϵ, we need at least $(RG/\epsilon)^2 = O(1/\epsilon^2)$ steps
- (quite slow)

Exercise

Support vector machines

- Let $\mathcal{D} := \{(x_i, y_i) \mid x_i \in \mathbb{R}^n, y_i \in \{\pm 1\}\}$
- We wish to find $w \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that

$$\min_{w,b} \quad \tfrac{1}{2}\|w\|_2^2 + C \sum_{i=1}^{m} \max[0, 1 - y_i(w^T x_i + b)]$$

- Derive and implement a subgradient method
- Plot evolution of objective function
- Experiment with different values of $C > 0$
- Plot and keep track of $f_{\text{min}}^k := \min_{0 \le t \le k} f(x^t)$

Polyak's stepsize

- Assume f^* is known (or can be estimated). Then use

$$\alpha_t = \frac{f^t - f^*}{\|g^t\|_2^2}$$

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- Motivation: recall bound

$$\|x^{t+1} - x^*\|^2 \leq \|x^t - x^*\|^2 - 2\alpha_t(f^t - f^*) + \alpha_t^2\|g^t\|^2$$

and minimize RHS.

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  and minimize RHS.
- Let's plug in α_t:

$$\|x^{t+1} - x^*\|^2 \leq \|x^t - x^*\|^2 - \frac{(f^t - f^*)^2}{\|g_t\|^2}$$

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- **Observation 1** $\|x^t - x^*\|$ decreases

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- **Observation 1** $\|x^t - x^*\|$ decreases
- Recursion:

$$\sum_{t=1}^{k} \frac{(f^t - f^*)^2}{\|g_t\|^2} \quad \leq \quad \|x^1 - x^*\|^2 \quad \leq \quad R^2$$

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- Now use $\|g^t\| \leq G$

$$\sum_{t=1}^{k} (f^t - f^*)^2 \leq R^2 G^2$$

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- **Observation 2** $f^t \to f^*$

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- Now use $\|g^t\| \leq G$

$$\sum_{t=1}^{k} (f^t - f^*)^2 \leq R^2 G^2$$

- **Observation 2** $f^t \to f^*$
- for accuracy ε, need $k = (RG/\varepsilon)^2$

Constrained optimization

$$\min \quad f(x) \qquad \text{s.t.} \quad x \in \mathcal{C}$$

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$$\min \quad f(x) \qquad \text{s.t.} \quad x \in \mathcal{C}$$

- Previously:

$$x^{t+1} = x^t - \alpha_t g^t$$

- This could be infeasible!
  Solution: projection

Projected subgradient method

$$x^{k+1} = P_{\mathcal{C}}(x^k - \alpha_k g^k)$$

where $g^k \in \partial f(x^k)$ is any subgradient

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- **Projection** closest feasible point

$$P_{\mathcal{C}}(x) = \arg\min_{y \in \mathcal{C}} \|x - y\|^2$$

(Assume $\mathcal{C}$ is closed and convex, then projection is unique)

Projected subgradient method

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- **Projection** closest feasible point

$$P_{\mathcal{C}}(x) = \arg\min_{y \in \mathcal{C}} \|x - y\|^2$$

(Assume $\mathcal{C}$ is closed and convex, then projection is unique)

- Great as long as projection is "easy"
- Same questions as before:
  - Does it converge?
  - For which stepsizes?
  - How fast?

Convergence

Assumptions

- Min is attained: $f^* := \inf_x f(x) > -\infty$, with $f(x^*) = f^*$
- Bounded subgradients: $\|g\|_2 \leq G$ for all $g \in \partial f$
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Analysis

- Let $z^{t+1} = x^t - \alpha_t g^t$.
- Then $x^{t+1} = P_{\mathcal{C}}(z^{t+1})$.

Convergence

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- Min is attained: $f^* := \inf_x f(x) > -\infty$, with $f(x^*) = f^*$
- Bounded subgradients: $\|g\|_2 \leq G$ for all $g \in \partial f$
- Bounded domain: $\|x^0 - x^*\|_2 \leq R$

Analysis

- Let $z^{t+1} = x^t - \alpha_t g^t$.
- Then $x^{t+1} = P_{\mathcal{C}}(z^{t+1})$.
- Recall analysis of unconstrained method:

$$\begin{aligned} \|z^{t+1} - x^*\|_2^2 &= \|x^t - \alpha_t g^t - x^*\|_2^2 \\ &\leq \|x^t - x^*\|_2^2 + \alpha_t^2 \|g^t\|_2^2 - 2\alpha_t(f(x^t) - f^*) \\ &\cdots \end{aligned}$$

- Need to relate to $\|x^{t+1} - x^*\|_2^2$, the rest of the proof is the same as above.

Projection Theorem

Let $\mathcal{C}$ be nonempty, closed and convex.

- Optimality conditions: $y^* = P_{\mathcal{C}}(z)$ iff

$$\langle z - y^*, y - y^* \rangle \leq 0 \text{ for all } y \in \mathcal{C}$$

- The projection is nonexpansive:

$$\|P_{\mathcal{C}}(x) - P_{\mathcal{C}}(z)\| \;\leq\; \|x - z\|^2 \quad \text{for all } x, z \in \mathbb{R}^n.$$

Convergence

- Use nonexpansiveness of projection:

$$
\begin{aligned}
&\|x^t - \alpha_t g^t - x^*\|_2^2 \\
\leq\ &\|x^t - x^*\|_2^2 + \alpha_t^2 \|g^t\|_2^2 - 2\alpha_t (f(x^t) - f^*) \\
\ldots
\end{aligned}
$$

Convergence

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$$\begin{aligned}\|x^{t+1} - x^*\|_2^2 &= \|P_{\mathcal{C}}(x^t - \alpha_t g^t) - x^*\|_2^2 \\ &\leq \|x^t - \alpha_t g^t - x^*\|_2^2 \\ &\leq \|x^t - x^*\|_2^2 + \alpha_t^2 \|g^t\|_2^2 - 2\alpha_t (f(x^t) - f^*) \\ &\ldots\end{aligned}$$

Same convergence results as in unconstrained case:

- within neighborhood of optimal for constant step size
- converges for diminishing non-summable

Examples

$$\begin{aligned} \min \quad & \tfrac{1}{2}\|Ax - b\|_2^2 + \lambda\|x\|_1 \\ \text{s.t.} \quad & x \in \mathcal{C} \end{aligned}$$

- **Nonnegativity** $x \geq 0$
  $P_{\mathcal{C}}(z) = [z]_+$
  Update step: $x^{k+1} = [x^k - \alpha_k(A^T(Ax^k - b) + \lambda\,\mathrm{sgn}(x^k))]_+$

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$$\begin{aligned} z^{k+1} &= x^k - \alpha_k(A^T(Ax^k - b) + \lambda \operatorname{sgn}(x^k)) \\ x_i^{k+1} &= \operatorname{sgn}(z_i^{k+1}) \min\{|z_i^{k+1}|, 1\} \end{aligned}$$

Examples

- **Linear equality constraints** $Ax = b$ ($A \in \mathbb{R}^{n\times m}$ has rank n)

$$\begin{aligned} P_{\mathcal{C}}(x) &= z - A^\top (AA^\top)^{-1}(Az - b) \\ &= (I - A^\top (A^\top A)^{-1} A) z + A^\top (AA^\top)^{-1} b \end{aligned}$$

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- **Simplex** $x^\top 1 = 1$ and $x \geq 0$
  more complex but doable, similarly ℓ_1-norm ball

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  - simple
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Mirror Descent

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  Mirror Descent
- Improvements using more information (heavy-ball, filtered subgradient, ...)
- Don't forget the dual!
  - may be more amenable to optimization
  - duality gap

What we did not cover

- ♠ Adaptive stepsize tricks
- ♠ Space dilation methods, quasi-Newton style subgrads
- ♠ Barrier subgradient method
- ♠ Sparse subgradient method
- ♠ Ellipsoid method, center of gravity, etc. as subgradient methods
- ♠ And many more

References

- ♠ S. Boyd, EE364b Slides
- ♠ Bertsekas, Nonlinear Programming